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High-order expansions of the Detweiler-Whiting singular field in Kerr spacetime

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In a previous paper, we computed expressions for the Detweiler-Whiting singular field of point scalar, electromagnetic and gravitational charges following a geodesic of the Schwarzschild spacetime. We now extend this to the case of equatorial orbits in Kerr spacetime, using coordinate and covariant approaches to compute expansions of the singular field in scalar, electromagnetic and gravitational cases. As an application, we give the calculation of previously unknown mode-sum regularisation parameters. We also propose a new application of high order approximations to the singular field, showing how they may be used to compute m-mode regularization parameters for use in the m-mode effective source approach to self-force calculations.

I. INTRODUCTION

The two-body problem in general relativity is a longstanding open problem going back to work by Einstein himself. With recent advances in gravitational wave detector technology, this age-old problem has been given a new lease of life. Some of the key sources expected to be seen by both space and ground based gravitational wave detectors are black hole binaries. Accurate models of black hole binaries are required for their successful detection by gravitational wave detectors. This development is today motivating numerical, analytical and experimental relativists to work together with the goal of producing models of the inspiral and merger of binary black hole systems.

In modelling black hole binaries, it is widely accepted that for the scenario of an extreme mass ratio inspiral (EMRI), the self-force approach is the model of choice. EMRIs are expected to be seen by space-based detectors such as NGO/eLISA [1]. Although NGO/eLISA has been recently postponed, the gravitational wave research community are confident in its inevitable flight. In the meantime, recent research has shown the applicability of self-force calculations to other binary black hole configurations [2, 3], extending the application of self-force to ground-based detectors such as LIGO and VIRGO.

Within the self-force approach, one perturbatively solves for the motion of a small body in the background of a massive black hole. Formal derivations of the equations of motion of a small body moving in a curved spacetime have settled on the idea of a well-defined singular-regular split of the retarded field generated by the body [4–12]. Several practical self-force computation strategies have developed from these formal derivations, all of which are based on the now-justified assumption that the use of a distributional source is acceptable at first perturbative order. These strategies broadly fall into three categories: the *mode-sum* approach [13, 14], the *effective source* approach [15, 16] and *Green function* approaches [17, 18]. The key to all three approaches is the subtraction of an appropriate *singular* component from the retarded field to leave a finite *regular* field which is solely responsible for the self-force. This singular component must have the same singular structure as the full retarded field in the vicinity of the body and must not contribute to the self-force (or its contribution must be well known and can be corrected for). There are many choices for a singular field that satisfies these criteria, although not all choices are equal. Detweiler and Whiting [19] identified a particularly appropriate choice. Through a Green function decomposition, they defined a singular field which not only satisfies the above two criteria, but also has the property that when it is subtracted from the full retarded field, it leaves a regularized field which is a solution to the *homogeneous* wave equation. Extensions of this idea of a singular-regular split to extended charge distributions [8, 9], to second perturbative order [20–24] and fully non-perturbative contexts [25] have recently been developed.

In a previous paper [26] (from now on referred to as Paper I), we focussed our calculations on the Schwarzschild spacetime representing a non-rotating black hole. Although this is a possible physical scenario, it is believed that a more astrophysically realistic or probable situation would be that of a Kerr or rotating black hole spacetime. One of the primary goals of the self-force community is, therefore, the successful calculation of the self-force in Kerr spacetime, with particular emphasis on the gravitational case. To this end, we now adapt our previous work [26] to the Kerr spacetime.

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In Paper I, we developed approaches to computing highly accurate approximations to the Detweiler-Whiting singular field of point scalar and electromagnetic charges as well as that of a point mass. This was achieved through high-order series expansions in a parameter ϵ, which acts as a measure of distance from the particle's world-line. We also derived explicit expressions for the case of geodesic motion in Schwarzschild spacetime. In this paper, we extend this analysis to the case of eccentric, equatorial orbits in the Kerr spacetime. We find that all of the methods developed in Paper I may be applied to the Kerr case with little modification. Nevertheless, the detailed expressions are significantly more complicated in the Kerr case. Since our method is largely the same as that used in Paper I, we direct the reader there for full details and give here only the expressions which differ.

As applications of our high-order expansions of the singular field, in Paper I we derived expression which may be used to improve the accuracy of both the mode-sum and effective source approaches. Since the effective source approach requires that the source be evaluated in an extended region around the world-line, numerical evaluation can be time consuming, in particular, when using high order expansions such as the ones produced in this paper and in Paper I. Existing calculations have settled on expansions of the singular field to $\mathcal{O}(\epsilon^2)$ to be a particular 'sweet spot' for these calculations - up to this order the increase in complexity of the singular field and corresponding effective source is rewarded with an increase in accuracy. However, expansions above this order may well slow the calculations down to such a degree that the extra orders offer more of an hindrance than a help. In this paper we propose one solution to this problem, allowing most of the benefit to be reaped from high-order expansions without the need for using increasingly complicated high-order expansions in numerical evolutions. This idea makes use of the m-mode scheme, developed by Barack and Golbourn [15], which decomposes the retarded field and effective source into azimuthal modes; the resulting conservation of axial symmetry makes the scheme well suited to the Kerr space-time. By carrying out m-mode effective source calculations with an effective source accurate to some order, say $\mathcal{O}(\epsilon^2)$, one can obtain numerical values for the m-modes of the self-force which converges polynomially with $1/m$. Our technique then makes use of our higher terms of the singular field (those above $\mathcal{O}(\epsilon^2)$) to obtain a faster convergence of this m-mode sum and hence assist in the production of highly accurate values of the self-force.

The layout of this paper is as follows. In Sec. II, we use coordinate expansions to derive high-order regularization parameters for use in the mode-sum method. In doing so, we give the next two previously unknown non-zero regularization parameters in scalar, electromagnetic and gravitational cases. In Sec. III we propose a new application of high-order coordinate expansions of the singular field, showing how they may be used to derive m-mode regularization parameters for use in the m-mode effective source approach. In Sec. IV we summarize our results and discuss further prospects for their application.

Throughout this paper, we use units in which $G = c = 1$ and adopt the sign conventions of [27]. We denote symmetrization of indices using parenthesis (e.g. (ab)), anti-symmetrization using square brackets (e.g. $[ab]$), and exclude indices from (anti-)symmetrization by surrounding them by vertical bars (e.g. $(a|b|c)$, $[a|b|c]$). We denote pairwise (anti-)symmetrization using an overbar, e.g. $R_{(\overline{ab}\,\overline{cd})} = \frac{1}{2}(R_{abcd} + R_{cdab})$. Capital letters are used to denote the spinorial/tensorial indices appropriate to the field being considered. In many of our calculations, we have several spacetime points to be considered. Our convention is that:

- the point x refers to the point where the field is evaluated
- the point $\bar{x}$ refers to an arbitrary point on the world-line

In computing expansions, we use ϵ as an expansion parameter to denote the fundamental scale of separation, so that $x - \bar{x} \approx \mathcal{O}(\epsilon)$. Where tensors are to be evaluated at these points, we decorate their indices appropriately using $\bar{\ }$, e.g. T^a and $T^{\bar{a}}$ refer to tensors at x and $\bar{x}$, respectively.

II. l-MODE REGULARIZATION

One of the most successful self-force computation approaches to the date is the *mode-sum* scheme of Barack and Ori [13, 14]; the majority of existing calculations are based on it in one form or another [28–51]. The basic idea is to decompose the retarded field into spherical harmonic modes which are continuous and finite in general for the scalar case and in Lorenz gauge for the electromagnetic and gravitational cases. The spherical symmetry of the Schwarzschild spacetime makes this decomposition into spherical harmonic modes a natural choice. In Kerr spacetime, despite there being more natural choices (such as a decomposition into spheroidal harmonics), a decomposition of the singular field into spherical harmonic modes has been shown to be of practical use in computing the scalar self-force [44, 52]. While similar approaches have yet to be attempted in electromagnetic or gravitational cases, it seems likely that they are at least possible in principle.

A key component of the mode-sum calculation involves the subtraction of the so-called *regularization parameters* - analytically derived expressions which render the formally divergent sum over spherical harmonic modes finite. In this

section, we derive these parameters from our singular field expressions and show how they may be used to compute the self-force with unprecedented accuracy.

A. Mode Sum Concept

The self-force for scaler, electromagnetic and gravitational cases, can be written generically as

$$F^a = p^a{}_A \varphi^A_{(\mathrm{R})} \tag{2.1}$$

where

$$\varphi^A_{(\mathrm{R})} = \varphi^A_{(\mathrm{ret})} - \varphi^A_{(\mathrm{S})} \tag{2.2}$$

is the regularised field and $p^a{}_A(x)$ is a tensor at x which depends on the type of charge. We can therefore rewrite the self-force as

$$F^a = p^a{}_A \varphi^A_{(\mathrm{ret})} - p^a{}_A \varphi^A_{(\mathrm{S})}. \tag{2.3}$$

Carrying out a spherical harmonic decomposition on the field,

$$\varphi^A_{(\mathrm{ret})/(\mathrm{S})} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \varphi^A_{lm\,(\mathrm{ret})/(\mathrm{S})} Y_{lm}(\theta,\phi), \tag{2.4}$$

allows the self-force to be rewritten as,

$$F^a = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(p^a{}_A \varphi^A_{lm\,(\mathrm{ret})} - p^a{}_A \varphi^A_{lm\,(\mathrm{S})} \right) Y_{lm}(\theta_0,\phi_0). \tag{2.5}$$

Defining the l component of the retarded or singular self-force to be,

$$F^a_{l\;(\mathrm{ret})/(\mathrm{S})} = p^a{}_A \sum_{m=-l}^{l} \varphi^A_{lm\,(\mathrm{ret})/(\mathrm{S})} Y_{lm}(\theta_0,\phi_0), \tag{2.6}$$

the self-force can be expressed as

$$F^a = \sum_{l=0}^{\infty} \left(F^a_{l\;(\mathrm{ret})} - F^a_{l\;(\mathrm{S})} \right). \tag{2.7}$$

It is the last term on the right that we calculate in this section for each of scalar, electromagnetic and gravitational cases in Kerr spacetime.

Our explicit expression for the l-modes of the singular self-force in Kerr spacetime is written as an expansion about the world-line point $\bar{x}$, that is

$$\begin{aligned} F^a_{l\;(\mathrm{S})} = {} & F^l_{a[\text{-}1]}\left(r_0,t_0\right)\epsilon^{-2} + F^l_{a[0]}\left(r_0,t_0\right)\epsilon^{-1} + F^l_{a[2]}\left(r_0,t_0\right)\epsilon^{1} \\ & + F^l_{a[4]}\left(r_0,t_0\right)\epsilon^{3} + F^l_{a[6]}\left(r_0,t_0\right)\epsilon^{5} + \dots, \end{aligned} \tag{2.8}$$

where we are missing even powers of ϵ above -2, as these are zero - this will be shown to be the case later in this section. When summed over l, the contribution of $F^a_{l[2]}\left(r_0,t_0\right)$, and higher terms, to the self-force is zero. However, if we ignore these higher terms in the approximation of $\varphi^A_{lm\,(\mathrm{S})}$, then the approximation for $\varphi^A_{lm\,(\mathrm{R})}$ is only C^1, causing the sum over l to be polynomially, rather than exponentially convergent in $1/l$. Therefore, despite these terms having zero contribution to the self-force, when it comes to numerically calculating the self-force using a finite number of l modes, the inclusion of the higher order terms dramatically reduces the number of modes required and hence computation time. For this reason, every extra term or regularisation parameter that can be calculated is important.

B. Rotated Coordinates

In order to obtain expressions which are readily written as mode-sums, previous calculations [13, 30, 32] found it useful to work in a rotated coordinate frame. In Paper I, we found it most efficient to carry out this rotation prior to doing any calculations; this also holds in the Kerr case. To this end, we introduce coordinates on the 2-sphere at $\bar{x}$ in the form

$$w_1 = 2\sin\left(\frac{\alpha}{2}\right)\cos\beta, \qquad w_2 = 2\sin\left(\frac{\alpha}{2}\right)\sin\beta, \tag{2.9}$$

where α and β are rotated angular coordinates given by

$$\sin\theta\cos\phi = \cos\alpha, \tag{2.10}$$
$$\sin\theta\sin\phi = \sin\alpha\cos\beta, \tag{2.11}$$
$$\cos\theta = \sin\beta\sin\alpha. \tag{2.12}$$

The Kerr metric in these coordinates (with $\bar{x}$ chosen to lie on the equator, ie. $\alpha = 0$) is given by the line element

$$\begin{aligned}
ds^2 = {} & \left[\frac{8Mr}{4r^2 + a^2 w_2^2\left(4 - w_1^2 - w_2^2\right)} - 1\right] dt^2 + \left[\frac{4r^2 + a^2 w_2^2\left(4 - w_1^2 - w_2^2\right)}{4\left(r^2 - 2Mr + a^2\right)}\right] dr^2 \\
& + 2dtdw_1 \frac{-2aMr\left[8 - w_2^2\left(6 - w_1^2 - w_2^2\right)\right]}{\sqrt{4 - w_1^2 - w_2^2}\left[4r^2 + a^2 w_2^2\left(4 - w_1^2 - w_2^2\right)\right]} + 2dtdw_2 \frac{2aMrw_1w_2\left(6 - w_1^2 - w_2^2\right)}{\sqrt{4 - w_1^2 - w_2^2}\left[4r^2 + a^2 w_2^2\left(4 - w_1^2 - w_2^2\right)\right]} \\
& + \frac{1}{4\left(4 - w_1^2 - w_2^2\right)\left[4 - w_2^2\left(4 - w_1^2 - w_2^2\right)\right]}\left[g_{w_1w_1} dw_1^2 + 2g_{w_1w_2} dw_1 dw_2 + g_{w_2w_2} dw_2^2\right],
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
g_{w_1w_1} = {} & w_1^2 w_2^2\left[4r^2 + a^2 w_2^2\left(4 - w_1^2 - w_2^2\right)\right] + \left[8 - w_2^2\left(6 - w_1^2 - w_2^2\right)\right]^2\left[r^2 + a^2 + 2Ma^2 r\frac{4 - w_2^2\left(4 - w_1^2 - w_2^2\right)}{4r^2 + a^2 w_2^2\left(4 - w_1^2 - w_2^2\right)}\right], \\
g_{w_1w_2} = {} & w_1 w_2\left(w_1^2 + 2w_2^2 - 4\right)\left[4r^2 + a^2 w_2^2(4 - w_1^2 - w_2^2)\right] \\
& + w_1 w_2\left(6 - w_1^2 - w_2^2\right)\left[8 - w_2^2\left(6 - w_1^2 - w_2^2\right)\right]\left[r^2 + a^2 + 2Ma^2 r\frac{4 - w_2^2\left(4 - w_1^2 - w_2^2\right)}{4r^2 + a^2 w_2^2\left(4 - w_1^2 - w_2^2\right)}\right], \\
g_{w_2w_2} = {} & \left(4 - w_1^2 - w_2^2\right)^2\left[4r^2 + a^2 w_2^2(4 - w_1^2 - w_2^2)\right] \\
& + w_1^2 w_2^2\left(6 - w_1^2 - w_2^2\right)^2\left[r^2 + a^2 + 2Ma^2 r\frac{4 - w_2^2\left(4 - w_1^2 - w_2^2\right)}{4r^2 + a^2 w_2^2\left(4 - w_1^2 - w_2^2\right)}\right].
\end{aligned} \tag{2.14}$$

As in the Schwarzschild case, this algebraic form has an advantage over its trigonometric counterpart in computer algebra programmes where trigonometric functions tend to slow calculations down. Despite the apparent complexity of the Kerr metric in this form, calculations of the regularization parameters are more efficient than using Boyer-Lindquist coordinates and rotating the resulting complicated expressions.

C. Mode Decomposition

Having calculated the singular field using the Kerr metric in the above form and using the methods described in Paper I, it is straightforward to calculate the singular component of the self-force, F^a, for scalar, electromagnetic and gravitational cases using Eq. (2.1) with the singular field substituted for the regular field. We obtain a multipole decomposition of F^a by writing

$$F^a\left(r, t, \alpha, \beta\right) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} F^a_{lm}\left(r, t\right) Y_{lm}\left(\alpha, \beta\right), \tag{2.15}$$

where $Y^{lm}\left(\theta, \phi\right)$ are scalar spherical harmonics, and accordingly

$$F^a_{lm}\left(r, t\right) = \int F^a\left(r, t, \alpha, \beta\right) Y^*_{lm}\left(\alpha, \beta\right) d\Omega. \tag{2.16}$$

To calculate the l-mode contribution at $\bar{x} = (t_0, r_0, \alpha_0, \beta_0)$, we have

$$F_l^a(r_0, t_0) = \lim_{\Delta r\to 0} \sum_{m=-l}^{l} F_{lm}^a(r_0 + \Delta r, t_0)\, Y_{lm}(\alpha_0, \beta_0)\,. \tag{2.17}$$

With the particle on the pole in the rotated coordinate system, $Y_{lm}(\alpha_0 = 0, \beta_0) = 0$ for all $m \neq 0$. This also allows us, without loss of generality, to take $\beta_0 = 0$. Taking α_0, β_0 and m all to be equal to zero in Eq. (2.17) gives

$$\begin{aligned} F_l^a(r_0, t_0) &= \lim_{\Delta r\to 0} \sqrt{\frac{2l+1}{4\pi}} F_{l0}^a(r_0 + \Delta r, t_0) \\ &= \frac{2l+1}{4\pi} \lim_{\Delta r\to 0} \int F^a(r_0 + \Delta r, t_0, \alpha, \beta)\, P_l(\cos\alpha)\, d\Omega. \end{aligned} \tag{2.18}$$

Using the methods of Paper I, a coordinate expansion of the singular self-force $F^a(r, t, \alpha, \beta)$ may be written in the form

$$F^a(r, t, \alpha, \beta) = \sum_{n=1}^{\infty} \frac{B^{a(3n-2)}}{\rho^{2n+1}} \epsilon^{n-3}, \tag{2.19}$$

where $B^{a(k)} = b^a_{a_1 a_2 \ldots a_k}(\bar{x}) \Delta x^{a_1} \Delta x^{a_2} \ldots \Delta x^{a_k}$. In using Eq. (2.19) to determine the regularization parameters, we only need to take the sum to the appropriate order: $n = 1$ for $F^a_{l[-1]}$, $n = 2$ for $F^a_{l[0]}$, etc. Explicitly, in our rotated coordinates

$$\begin{aligned} \rho(r, t, \alpha, \beta)^2 =& \frac{\Delta r^2 r_0 \left[r_0\left(a^2E^2 - L^2\right) + 2M(L - aE)^2 + E^2 r_0^3\right]}{\left(a^2 - 2Mr_0 + r_0^2\right)^2} + \Delta t\left[\Delta w_1\Big(-\frac{4aM}{r_0} - 2EL\Big) - \frac{2\Delta r E r_0^2 \dot{r}_0}{a^2 - 2Mr_0 + r_0^2}\right] \\ &+ \Delta w_1^2\left(\frac{2a^2M}{r_0} + a^2 + L^2 + r_0^2\right) + \frac{2\Delta r \Delta w_1 L r_0^2 \dot{r}_0}{a^2 - 2Mr_0 + r_0^2} + \Delta t^2\left(E^2 + \frac{2M}{r_0} - 1\right) + \Delta w_2^2 r_0^2, \end{aligned} \tag{2.20}$$

where the α, β dependence is contained exclusively in Δw_1 and Δw_2. Here, $E = -u_t$ and $L = u_\phi$ are the energy per unit mass and angular momentum along the axis of symmetry, respectively. In particular, taking $t = t_0$ ($\Delta t = 0$) allows us to write

$$\begin{aligned} \rho(r, t_0, \alpha, \beta)^2 =& \frac{\Delta r^2 r_0 \left[E r_0\left(a^2 + r_0^2\right) + 2aM(aE - L)\right]^2}{\left(a^2 - 2Mr_0 + r_0^2\right)^2\left[r_0\left(a^2 + L^2\right) + 2a^2M + r_0^3\right]} + \Delta w_2^2 r_0^2 \\ &+ \left(\frac{2a^2M}{r_0} + a^2 + L^2 + r_0^2\right)\left[\Delta w_1 + \frac{\Delta r L r_0^3 \dot{r}_0}{\left(a^2 - 2Mr_0 + r_0^2\right)\left(2a^2M + a^2 r_0 + L^2 r_0 + r_0^3\right)}\right]^2. \end{aligned} \tag{2.21}$$

For the mode-sum decomposition, it is favourable to work with $\rho_0(\alpha, \beta)^2 \equiv \rho(r_0, t_0, \alpha, \beta)^2$ in the form

$$\rho_0(\alpha, \beta)^2 = 2(1 - \cos\alpha)\,\zeta^2\left(1 - k\sin\beta^2\right) \tag{2.22}$$

This can be achieved by rewriting Eq. (2.21) with $\Delta r \to 0$ as

$$\rho_0(\alpha, \beta)^2 = \zeta^2 \Delta w_1^2 + r_0^2 \Delta w_2^2, \tag{2.23}$$

where

$$\zeta^2 = L^2 + r_0^2 + \frac{2a^2M}{r_0} + a^2 \tag{2.24}$$

and rearranging to give

$$\rho_0(\alpha, \beta)^2 = 2(1 - \cos\alpha)\,\zeta^2\left[1 - \left(\frac{\zeta^2 - r_0^2}{\zeta^2}\right)\sin\beta^2\right]. \tag{2.25}$$

which is equivalent to Eq. (2.22) with $k = \frac{\zeta^2 - r_0^2}{\zeta^2}$. Defining $\chi(\beta) \equiv 1 - k\sin^2\beta$, we can rewrite our Δw's in the alternate forms

$$\Delta w_1^2 = 2(1 - \cos\alpha)\cos^2\beta = \frac{{\rho_0}^2}{\zeta^2\chi}\cos^2\beta = \frac{{\rho_0}^2}{\left(\zeta^2 - r_0^2\right)\chi}\left(k - (1 - \chi)\right), \tag{2.26}$$

$$\Delta w_2^2 = 2(1 - \cos\alpha)\sin^2\beta = \frac{{\rho_0}^2}{\zeta^2\chi}\sin^2\beta = \frac{{\rho_0}^2}{\left(\zeta^2 - r_0^2\right)\chi}(1 - \chi). \tag{2.27}$$

Suppose, for the moment, that we may take the limit in Eq. (2.18) through the integral sign, then using our alternate forms we have

$$\lim_{\Delta r\to 0}\frac{B^{a(3n-2)}}{\rho^{2n+1}}\epsilon^{n-3}=\frac{b^a_{i_1 i_2\ldots i_{3n-2}}(r_0)\Delta w^{i_1}\Delta w^{i_2}\ldots\Delta w^{i_{3n-2}}}{{\rho_0}^{2n+1}}\epsilon^{n-3}={\rho_0}^{n-3}\epsilon^{n-3}c^a_{(n)}(r_0,\chi). \tag{2.28}$$

In [53], it was shown that the integral and limit in Eq. (2.18) are indeed interchangeable for all orders except the leading order, $n=1$ term, where the limiting $1/{\rho_0}^3$ would not be integrable. Thus we find the singular self-force now has the form

$$\begin{aligned}F^a_l\left(r_0,t_0\right)=&\frac{2l+1}{4\pi}\left[\epsilon^{-2}\lim_{\Delta r\to 0}\int\frac{B^{a(1)}\left(r,t_0,\alpha,\beta\right)}{\rho^3\left(r,t_0,\alpha,\beta\right)}P_l\left(\cos\alpha\right)d\Omega+\sum_{n=2}^{\infty}\epsilon^{n-3}\int{\rho_0}^{n-3}c^a_{(n)}\left(r_0,\chi\right)P_l\left(\cos\alpha\right)d\Omega\right]\\ \equiv&F^a_{l[\text{-}1]}\left(r_0,t_0\right)\epsilon^{-2}+F^a_{l[0]}\left(r_0,t_0\right)\epsilon^{-1}+F^a_{l[2]}\left(r_0,t_0\right)\epsilon^{1}+F^a_{l[4]}\left(r_0,t_0\right)\epsilon^{3}+F^a_{l[6]}\left(r_0,t_0\right)\epsilon^{5}+\ldots,\end{aligned} \tag{2.29}$$

where the β dependence in the $c^a_{(n)}$'s are hidden in χ, while the α and β dependence of $F^a\left(r,t_0,\alpha,\beta\right)$ is hidden in both the ρ's and $c^a_{(n)}$'s. Note here that we use the convention that a subscript in square brackets denotes the term which will contribute at that order in $1/l$. Furthermore, the integrand in the summation is odd or even under $\Delta w_i\to-\Delta w_i$ according to whether n (and so $3n-2$) is odd or even. As a result only the even terms are non-vanishing, while $F^a_{l[1]}\left(r_0,t_0\right)=F^a_{l[3]}\left(r_0,t_0\right)=F^a_{l[5]}\left(r_0,t_0\right)=0$, etc.

Some care is required in order to obtain easily integrable expressions in the case of eccentric orbits. We use the approach of previous methods [30, 32, 53, 54] (and also employed in Paper I) by redefining our Δw_1 coordinate in such a way that the cross-terms involving $\Delta r\Delta w_1$ in ρ_0 vanish. That is, we make the replacement $\Delta w_1\to\Delta w_1+c\Delta r$, where c is given by

$$c=\frac{-Lr_0^3\dot{r}_0}{\left(a^2-2Mr_0+r_0^2\right)\left(2a^2M+a^2r_0+L^2r_0+r_0^3\right)}. \tag{2.30}$$

This allows us to write

$$\begin{aligned}\rho\left(r,t_0,\alpha,\beta\right)^2&=\nu^2\Delta r^2+\zeta^2\Delta w_1^2+r_0^2\Delta w_2^2\\&=\nu^2\Delta r^2+2\chi\zeta^2\left(1-\cos\alpha\right)\end{aligned} \tag{2.31}$$

where ν is an expression involving r_0, a, E and L. This can be easily rearranged to give

$$\begin{aligned}\rho\left(r,t_0,\alpha,\beta\right)^{-3}&=\zeta^{-3}\left(2\chi\right)^{-3/2}\left(\delta^2+1-\cos\alpha\right)^{-3/2}\\&=\zeta^{-3}\left(2\chi\right)^{-3/2}\sum_{l=0}\mathcal{A}_l^{-3/2}(\delta)P_l\left(\cos\alpha\right),\end{aligned} \tag{2.32}$$

where

$$\delta^2=\frac{\nu^2\Delta r^2}{2\zeta^2\chi}\qquad\text{and}\qquad\mathcal{A}_l^{-\frac{3}{2}}(\delta)=\frac{2l+1}{\delta} \tag{2.33}$$

Here $\mathcal{A}_l^{-\frac{3}{2}}(\delta)$ is derived from the generating function of the Legendre polynomials as shown in Eq. (D12) of [30]. We can now express $\rho\left(r,t_0,\alpha,\beta\right)^{-3}$ as

$$\rho\left(r,t_0,\alpha,\beta\right)^{-3}=\frac{1}{\zeta^2\nu\chi\sqrt{\Delta r^2}}\sum_{l=0}\left(l+\tfrac{1}{2}\right)P_l\left(\cos\alpha\right) \tag{2.34}$$

Bringing this result into our expression for $F^a_{l[\text{-}1]}\left(r_0,t_0\right)$ from Eq. (2.29) and integrating over α gives

$$\begin{aligned}F^a_{l[\text{-}1]}\left(r_0,t_0\right)&=\frac{1}{2\pi}\left(l+\tfrac{1}{2}\right)\lim_{\Delta r\to 0}\frac{1}{\zeta^2\nu\sqrt{\Delta r^2}}\int\frac{\tilde{B}^{a(1)}P_l\left(\cos\alpha\right)}{\chi}d\Omega\\&=\left(l+\tfrac{1}{2}\right)\lim_{\Delta r\to 0}\frac{\tilde{b}^a_{a_r}\Delta r}{\zeta^2\nu\sqrt{\Delta r^2}}\frac{1}{2\pi}\int\chi^{-1}d\beta\\&=\left(l+\tfrac{1}{2}\right)\frac{\tilde{b}^a_{a_r}\,\text{sgn}\left(\Delta r\right)}{\zeta\nu r_0}\end{aligned} \tag{2.35}$$

where the first equality takes advantage of the orthogonal nature of the $P_l(\cos\alpha)$ and the last equality comes from taking the limit as $\Delta r \to 0$ and noting from Appendix C of [30] that the integral is a special case of the hypergeometric functions given by

$$\frac{1}{2\pi}\int \chi^{-1} d\beta = F\left(1, \tfrac{1}{2}; 1; k\right) = \frac{1}{\sqrt{1-k}} = \frac{\zeta}{r_0}. \tag{2.36}$$

$B^{a(1)}$ and $b^a_{a_r}$ also now carry a tilde to signify that they are not the exact same $B^{a(1)}$ and $b^a_{a_r}$ from Eq. (2.19); the tilde reflects the fact that they have also undergone the coordinate shift $\Delta w_1 \to \Delta w_1 + c\Delta r$.

In the higher order terms in Eq. (2.29), we may immediately work with ${\rho_0}^2 = 2\chi\zeta^2(1-\cos\alpha)$ so,

$$\begin{aligned}\rho_0\left(r_0, t_0, \alpha, \beta\right)^n &= \zeta^n\left[2\chi\left(1-\cos\alpha\right)\right]^{n/2} \\ &= \zeta^n\left(2\chi\right)^{n/2}\sum_{l=0}\mathcal{A}_l^{n/2}(0)P_l\left(\cos\alpha\right),\end{aligned} \tag{2.37}$$

where $\mathcal{A}_l^{-\frac{1}{2}}(0) = \sqrt{2}$ from the generating function of the Legendre polynomials and, as given in Appendix D of [30], for $(n+1)/2 \in \mathbb{N}$

$$\mathcal{A}_l^{n/2}(0) = \frac{\mathcal{P}_{n/2}\left(2l+1\right)}{\left(2l-n\right)\left(2l-n+2\right)\ldots\left(2l+n\right)\left(2l+n+2\right)}, \tag{2.38}$$

$$\tag{2.39}$$

where

$$\mathcal{P}_{n/2} = (-1)^{(n+1)/2}\, 2^{1+n/2}\left(n!!\right)^2. \tag{2.40}$$

In this case the angular integrals involve

$$\frac{1}{2\pi}\int\frac{d\beta}{\chi(\beta)^{n/2}} = \left\langle \chi^{-n/2}(\beta)\right\rangle = {}_2F_1\left(\frac{n}{2}, \frac{1}{2}, 1, k\right) \tag{2.41}$$

where $(n+1)/2 \in \mathbb{N}\cup\{0\}$. The resulting equations can then be tidied up using the following special cases of hypergeometric functions

$$\left\langle \chi^{-\frac{1}{2}}\right\rangle = \mathcal{F}_{\frac{1}{2}}(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k\right) = \frac{2}{\pi}\mathcal{K}, \tag{2.42}$$

$$\left\langle \chi^{\frac{1}{2}}\right\rangle = \mathcal{F}_{-\frac{1}{2}}(k) = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k\right) = \frac{2}{\pi}\mathcal{E}, \tag{2.43}$$

where

$$\mathcal{K} \equiv \int_0^{\pi/2}(1-k\sin^2\beta)^{-1/2}d\beta, \quad \mathcal{E} \equiv \int_0^{\pi/2}(1-k\sin^2\beta)^{1/2}d\beta \tag{2.44}$$

are complete elliptic integrals of the first and second kinds, respectively. All other powers of χ can be integrated to give hypergeometric functions that can then be manipulated to be one of the above by the use of the recurrence relation in Eq. (15.2.10) of [55],

$$\mathcal{F}_{p+1}(k) = \frac{p-1}{p\left(k-1\right)}\mathcal{F}_{p-1}(k) + \frac{1-2p+\left(p-\frac{1}{2}\right)k}{p\left(k-1\right)}\mathcal{F}_p(k). \tag{2.45}$$

In the next sections, we give the results of applying this calculation to each of scalar, electromagnetic and gravitational cases in turn. In doing so, we omit the explicit dependence on l which in each case is

$$\begin{gathered}F^a_{l[\text{-}1]} = (2l+1)F^a_{[\text{-}1]}, \quad F^a_{l[0]} = F^a_{[0]}, \quad F^a_{l[2]} = \frac{F^a_{[2]}}{(2l-1)(2l+3)}, \\ F^a_{l[4]} = \frac{F^a_{[4]}}{(2l-3)(2l-1)(2l+3)(2l+5)}, \\ F^a_{l[6]} = \frac{F^a_{[6]}}{(2l-5)(2l-3)(2l-1)(2l+3)(2l+5)(2l+7)}.\end{gathered} \tag{2.46}$$

D. Scalar l-mode Regularization Parameters

In the scalar case, the singular part of the self-force is give by,

$$F_a = \partial_a \Phi^{(\mathrm{S})}, \tag{2.47}$$

where $\Phi^{(\mathrm{S})}$ is the scalar singular field. The scalar regularization parameters are then given by,

$$\begin{aligned}
F_{t[\text{-1}]} &= \frac{r_0 \dot{r}_0 \mathrm{sgn}\Delta \mathrm{r}}{r_0\left(a^2+L^2\right)+2a^2M+r_0^3},\\
F_{r[\text{-1}]} &= -\frac{\mathrm{sgn}\Delta\mathrm{r}\left(Er_0\left(a^2+r_0^2\right)+2aM(aE-L)\right)}{\left(a^2-2Mr_0+r_0^2\right)\left(r_0\left(a^2+L^2\right)+2a^2M+r_0^3\right)},\\
F_{\theta[\text{-1}]} &= 0, \quad F_{\phi[\text{-1}]} = 0,
\end{aligned} \tag{2.48}$$

$$F_{t[0]} = \frac{\dot{r}_0}{\pi r_0^2\left(r_0^2+L^2+\frac{2a^2M}{r_0}+a^2\right)^{3/2}}\left(F^{\mathcal{E}}_{t[0]}\mathcal{E}+F^{\mathcal{K}}_{t[0]}\mathcal{K}\right), \tag{2.49}$$

where

$$\begin{aligned}
F^{\mathcal{E}}_{t[0]} &= 4aLM\left(4a^4M^2+2a^4Mr_0+2a^2L^2Mr_0-a^2Mr_0^3-a^2r_0^4-L^2r_0^4\right)\\
&\quad + E\big(-12a^6M^3-16a^6M^2r_0-7a^6Mr_0^2-a^6r_0^3-4a^4L^2M^2r_0-6a^4L^2Mr_0^2-2a^4L^2r_0^3-6a^4M^2r_0^3-5a^4Mr_0^4\\
&\qquad -a^4r_0^5+a^2L^4Mr_0^2-a^2L^4r_0^3-5a^2L^2Mr_0^4-3a^2L^2r_0^5-2L^4r_0^5\big),\\
F^{\mathcal{K}}_{t[0]} &= -2aLM\big(2a^4M^2-a^4Mr_0-a^4r_0^2-a^2L^2Mr_0-2a^2L^2r_0^2-2a^2Mr_0^3-2a^2r_0^4-L^4r_0^2-2L^2r_0^4\big)\\
&\quad + E\big(4a^6M^3+4a^6M^2r_0+a^6Mr_0^2-2a^4L^2M^2r_0-a^4L^2Mr_0^2+2a^4M^2r_0^3+a^4Mr_0^4-2a^2L^4Mr_0^2+a^2L^2Mr_0^4\\
&\qquad +a^2L^2r_0^5+L^4r_0^5\big),
\end{aligned}$$

$$F_{r[0]} = \frac{F^{\mathcal{E}}_{t[0]}\mathcal{E}+F^{\mathcal{K}}_{t[0]}\mathcal{K}}{\pi r_0^3\left(2a^2M+a^2r_0+L^2r_0\right)^2\left(r_0^2+L^2+\frac{2a^2M}{r_0}+a^2\right)^{3/2}\left(r_0^2-2Mr_0+a^2\right)}, \tag{2.50}$$

where

$$\begin{aligned}
F^{\mathcal{E}}_{r[0]} &= \big(-24a^8M^3r_0-32a^8M^2r_0^2-14a^8Mr_0^3-2a^8r_0^4+24a^6L^2M^4+12a^6L^2M^3r_0-30a^6L^2M^2r_0^2-27a^6L^2Mr_0^3\\
&\qquad -6a^6L^2r_0^4+48a^6M^4r_0^2+40a^6M^3r_0^3-16a^6M^2r_0^4-20a^6Mr_0^5-4a^6r_0^6+8a^4L^4M^3r_0-12a^4L^4Mr_0^3\\
&\qquad -6a^4L^4r_0^4+36a^4L^2M^3r_0^3+12a^4L^2M^2r_0^4-21a^4L^2Mr_0^5-9a^4L^2r_0^6+24a^4M^3r_0^5+8a^4M^2r_0^6-6a^4Mr_0^7\\
&\qquad -2a^4r_0^8-2a^2L^6M^2r_0^2+a^2L^6Mr_0^3-2a^2L^6r_0^4+6a^2L^4M^2r_0^4+a^2L^4Mr_0^5-6a^2L^4r_0^6+12a^2L^2M^2r_0^6\\
&\qquad -3a^2L^2r_0^8+2L^6Mr_0^5-L^6r_0^6+2L^4Mr_0^7-L^4r_0^8\big)\\
&\quad -2aELM\big(24a^6M^3+28a^6M^2r_0+10a^6Mr_0^2+a^6r_0^3+8a^4L^2M^2r_0+8a^4L^2Mr_0^2+2a^4L^2r_0^3-4a^4Mr_0^4-2a^4r_0^5\\
&\qquad -2a^2L^4Mr_0^2+a^2L^4r_0^3-2a^2L^2Mr_0^4-a^2L^2r_0^5-6a^2Mr_0^6-3a^2r_0^7+L^4r_0^5-3L^2r_0^7\big)\\
&\quad +E^2\left(2a^2M+a^2r_0+r_0^3\right)\big(12a^6M^3+16a^6M^2r_0+7a^6Mr_0^2+a^6r_0^3+4a^4L^2M^2r_0+6a^4L^2Mr_0^2+2a^4L^2r_0^3\\
&\qquad +6a^4M^2r_0^3+5a^4Mr_0^4+a^4r_0^5-a^2L^4Mr_0^2+a^2L^4r_0^3+5a^2L^2Mr_0^4+3a^2L^2r_0^5+2L^4r_0^5\big),\\
F^{\mathcal{K}}_{r[0]} &= \big(8a^8M^3r_0+8a^8M^2r_0^2+2a^8Mr_0^3-8a^6L^2M^4+4a^6L^2M^3r_0+12a^6L^2M^2r_0^2+4a^6L^2Mr_0^3-16a^6M^4r_0^2\\
&\qquad -8a^6M^3r_0^3+8a^6M^2r_0^4+4a^6Mr_0^5+4a^4L^4M^3r_0+8a^4L^4M^2r_0^2+2a^4L^4Mr_0^3+8a^4L^2M^3r_0^3+12a^4L^2M^2r_0^4\\
&\qquad +2a^4L^2Mr_0^5-a^4L^2r_0^6-8a^4M^3r_0^5+2a^4Mr_0^7+4a^2L^6M^2r_0^2+16a^2L^4M^2r_0^4-2a^2L^4r_0^6+4a^2L^2M^2r_0^6\\
&\qquad -a^2L^2r_0^8+2L^6Mr_0^5-L^6r_0^6+2L^4Mr_0^7-L^4r_0^8\big)\\
&\quad +2aELM\big(8a^6M^3+4a^6M^2r_0-2a^6Mr_0^2-a^6r_0^3-4a^4L^2M^2r_0-6a^4L^2Mr_0^2-2a^4L^2r_0^3-8a^4M^2r_0^3\\
&\qquad -12a^4Mr_0^4-4a^4r_0^5-4a^2L^4Mr_0^2-a^2L^4r_0^3-10a^2L^2Mr_0^4-5a^2L^2r_0^5-6a^2Mr_0^6-3a^2r_0^7-L^4r_0^5-3L^2r_0^7\big)\\
&\quad -E^2\left(2a^2M+a^2r_0+r_0^3\right)\big(4a^6M^3+4a^6M^2r_0+a^6Mr_0^2-2a^4L^2M^2r_0-a^4L^2Mr_0^2+2a^4M^2r_0^3+a^4Mr_0^4\\
&\qquad -2a^2L^4Mr_0^2+a^2L^2Mr_0^4+a^2L^2r_0^5+L^4r_0^5\big),
\end{aligned}$$

$$F_{\theta[0]} = 0, \tag{2.51}$$

$$F_{\phi[0]} = \frac{L\dot{r}_0}{\pi r_0 \left(2a^2M + a^2 r_0 + L^2 r_0\right)^2 \left(r_0^2 + L^2 + \frac{2a^2M}{r_0} + a^2\right)^{1/2}} \left(F^{\mathcal{E}}_{r[0]}\mathcal{E} + F^{\mathcal{K}}_{r[0]}\mathcal{K}\right), \tag{2.52}$$

where

$$\begin{aligned} F^{\mathcal{E}}_{r[0]} &= -2a^4M^2 - a^4Mr_0 - a^2L^2Mr_0 - 4a^2Mr_0^3 - a^2r_0^4 - L^2r_0^4, \\ F^{\mathcal{K}}_{r[0]} &= r_0^3\left(4a^2M + a^2r_0 + L^2r_0\right). \end{aligned}$$

The regularisation parameters $F_{a[2]}$ and $F_{a[4]}$ are too large for paper format and have instead been made available electronically [56]. For the reader to get an understanding of the form and size of these expressions, we have included $F_{r[2]}$ for a circular orbit. This is given by,

$$F_{r[2]} = \frac{1}{6\pi r_0^6}\sqrt{\frac{2a\sqrt{Mr_0^5} - 3Mr_0^3 + r_0^4}{\left(a\sqrt{M} + r_0^{3/2}\right)^2 \left(a^2 + r_0(r_0 - 2M)\right)}} \tag{2.53}$$

$$\times \frac{\left(2a\sqrt{M} + \sqrt{r_0}(r_0 - 3M)\right)^{-2}\left(F^{\mathcal{E}}_{r[2]}\mathcal{E} + F^{\mathcal{K}}_{r[2]}\mathcal{K}\right)}{\left(a^4M + 2a^3\sqrt{Mr_0^3} + a^2r_0\left(-2M^2 + Mr_0 + r_0^2\right) - 4aM^{3/2}r_0^{5/2} + Mr_0^4\right)^3}, \tag{2.54}$$

where

$$\begin{aligned} F^{\mathcal{E}}_{r[2]} = {} & -\left(r_0^{3/2} + a\sqrt{M}\right)^2 \big(3M^3r_0^{33/2} - 36M^4r_0^{31/2} + 9a^2M^2r_0^{31/2} - 78aM^{7/2}r_0^{15} + 93M^5r_0^{29/2} + 27a^2M^3r_0^{29/2} \\ & + 9a^4Mr_0^{29/2} + 744aM^{9/2}r_0^{14} + 44a^3M^{5/2}r_0^{14} + 3a^6r_0^{27/2} - 60M^6r_0^{27/2} - 1068a^2M^4r_0^{27/2} - 120a^4M^2r_0^{27/2} \\ & - 1890aM^{11/2}r_0^{13} + 60a^3M^{7/2}r_0^{13} + 18a^5M^{3/2}r_0^{13} + 3681a^2M^5r_0^{25/2} + 913a^4M^3r_0^{25/2} + 15a^6Mr_0^{25/2} \\ & + 1332aM^{13/2}r_0^{12} - 704a^3M^{9/2}r_0^{12} - 404a^5M^{5/2}r_0^{12} + 24a^7\sqrt{M}r_0^{12} + 6a^8r_0^{23/2} - 4251a^2M^6r_0^{23/2} \\ & - 3846a^4M^4r_0^{23/2} - 357a^6M^2r_0^{23/2} + 2472a^3M^{11/2}r_0^{11} + 3434a^5M^{7/2}r_0^{11} + 24a^7M^{3/2}r_0^{11} + 2178a^2M^7r_0^{21/2} \\ & + 4559a^4M^5r_0^{21/2} + 732a^6M^3r_0^{21/2} + 81a^8Mr_0^{21/2} - 3336a^3M^{13/2}r_0^{10} - 7502a^5M^{9/2}r_0^{10} - 1328a^7M^{5/2}r_0^{10} \\ & + 48a^9\sqrt{M}r_0^{10} - 786a^4M^6r_0^{19/2} + 1421a^6M^4r_0^{19/2} - 366a^8M^2r_0^{19/2} - 1272a^3M^{15/2}r_0^9 + 5324a^5M^{11/2}r_0^9 \\ & + 4322a^7M^{7/2}r_0^9 + 174a^9M^{3/2}r_0^9 + 3501a^4M^7r_0^{17/2} - 5631a^6M^5r_0^{17/2} - 1651a^8M^3r_0^{17/2} + 156a^{10}Mr_0^{17/2} \\ & + 942a^5M^{13/2}r_0^8 - 10a^7M^{9/2}r_0^8 - 1360a^9M^{5/2}r_0^8 - 3798a^4M^8r_0^{15/2} - 5611a^6M^6r_0^{15/2} + 4900a^8M^4r_0^{15/2} \\ & + 288a^{10}M^2r_0^{15/2} + 6648a^5M^{15/2}r_0^7 - 2186a^7M^{11/2}r_0^7 - 1512a^9M^{7/2}r_0^7 + 264a^{11}M^{3/2}r_0^7 + 4740a^6M^7r_0^{13/2} \\ & + 5292a^8M^5r_0^{13/2} - 1686a^{10}M^3r_0^{13/2} - 2088a^5M^{17/2}r_0^6 - 11734a^7M^{13/2}r_0^6 + 2090a^9M^{9/2}r_0^6 \\ & + 384a^{11}M^{5/2}r_0^6 + 2844a^6M^8r_0^{11/2} - 2100a^8M^6r_0^{11/2} - 2490a^{10}M^4r_0^{11/2} + 246a^{12}M^2r_0^{11/2} \\ & + 3528a^7M^{15/2}r_0^5 + 8106a^9M^{11/2}r_0^5 - 790a^{11}M^{7/2}r_0^5 - 348a^6M^9r_0^{9/2} - 5398a^8M^7r_0^{9/2} + 495a^{10}M^5r_0^{9/2} \\ & + 414a^{12}M^3r_0^{9/2} + 360a^7M^{17/2}r_0^4 - 2174a^9M^{13/2}r_0^4 - 2598a^{11}M^{9/2}r_0^4 + 120a^{13}M^{5/2}r_0^4 + 708a^8M^8r_0^{7/2} \\ & + 3844a^{10}M^6r_0^{7/2} - 126a^{12}M^4r_0^{7/2} - 728a^9M^{15/2}r_0^3 + 570a^{11}M^{11/2}r_0^3 + 324a^{13}M^{7/2}r_0^3 - 543a^{10}M^7r_0^{5/2} \\ & - 1216a^{12}M^5r_0^{5/2} + 24a^{14}M^3r_0^{5/2} + 554a^{11}M^{13/2}r_0^2 - 52a^{13}M^{9/2}r_0^2 + 186a^{12}M^6r_0^{3/2} + 144a^{14}M^4r_0^{3/2} \\ & - 188a^{13}M^{11/2}r_0 - 24a^{14}M^5\sqrt{r_0} + 24a^{15}M^{9/2}\big), \\ F^{\mathcal{K}}_{r[2]} = {} & r_0^{7/2}\left(r_0^{3/2} - 3M\sqrt{r_0} + 2a\sqrt{M}\right)\big(3M^3r_0^{29/2} - 24M^4r_0^{27/2} + 9a^2M^2r_0^{27/2} - 78aM^{7/2}r_0^{13} + 33M^5r_0^{25/2} \\ & + 57a^2M^3r_0^{25/2} + 9a^4Mr_0^{25/2} + 486aM^{9/2}r_0^{12} + 44a^3M^{5/2}r_0^{12} + 3a^6r_0^{23/2} - 885a^2M^4r_0^{23/2} - 102a^4M^2r_0^{23/2} \\ & - 612aM^{11/2}r_0^{11} + 146a^3M^{7/2}r_0^{11} + 18a^5M^{3/2}r_0^{11} + 1845a^2M^5r_0^{21/2} + 688a^4M^3r_0^{21/2} + 9a^6Mr_0^{21/2} \\ & - 1086a^3M^{9/2}r_0^{10} - 354a^5M^{5/2}r_0^{10} + 24a^7\sqrt{M}r_0^{10} - 1206a^2M^6r_0^{19/2} - 1805a^4M^4r_0^{19/2} - 252a^6M^2r_0^{19/2} \end{aligned}$$

$$
\begin{aligned}
&+2580a^3M^{11/2}r_0^9+2546a^5M^{7/2}r_0^9+66a^7M^{3/2}r_0^9-528a^4M^5r_0^{17/2}-158a^6M^3r_0^{17/2}+78a^8Mr_0^{17/2}\\
&+12a^3M^{13/2}r_0^8-3176a^5M^{9/2}r_0^8-998a^7M^{5/2}r_0^8-189a^4M^6r_0^{15/2}+3058a^6M^4r_0^{15/2}+171a^8M^2r_0^{15/2}\\
&-24a^5M^{11/2}r_0^7+200a^7M^{7/2}r_0^7+132a^9M^{3/2}r_0^7+1143a^4M^7r_0^{13/2}-368a^6M^5r_0^{13/2}-1263a^8M^3r_0^{13/2}\\
&-1962a^5M^{13/2}r_0^6+1250a^7M^{9/2}r_0^6+216a^9M^{5/2}r_0^6-639a^6M^6r_0^{11/2}-347a^8M^4r_0^{11/2}+123a^{10}M^2r_0^{11/2}\\
&+684a^5M^{15/2}r_0^5+1978a^7M^{11/2}r_0^5-590a^9M^{7/2}r_0^5-879a^6M^7r_0^{9/2}+232a^8M^5r_0^{9/2}+186a^{10}M^3r_0^{9/2}\\
&-684a^7M^{13/2}r_0^4-832a^9M^{9/2}r_0^4+60a^{11}M^{5/2}r_0^4+114a^6M^8r_0^{7/2}+1048a^8M^6r_0^{7/2}-90a^{10}M^4r_0^{7/2}\\
&-108a^7M^{15/2}r_0^3+252a^9M^{11/2}r_0^3+144a^{11}M^{7/2}r_0^3-147a^8M^7r_0^{5/2}-452a^{10}M^5r_0^{5/2}+12a^{12}M^3r_0^{5/2}\\
&+146a^9M^{13/2}r_0^2-38a^{11}M^{9/2}r_0^2+69a^{10}M^6r_0^{3/2}+72a^{12}M^4r_0^{3/2}-70a^{11}M^{11/2}r_0-12a^{12}M^5\sqrt{r_0}\\
&+12a^{13}M^{9/2}\big).
\end{aligned}
$$

E. Electromagnetic l-mode Regularization Parameters

In the electromagnetic case, the singular part of the self-force is given by

$$F^a = e\, g^{ab} u^c A^{(\mathrm{S})}_{[c,b]}, \tag{2.55}$$

where e is the charge of the particle, g^{ab} is the inverse metric, u^c is the four-velocity and $A^{(\mathrm{S})}_{c,b}$ is the electromagnetic singular field. Here, an ambiguity arises in the definition of u^a in the angular directions *away* from the world-line. In Eq. (2.55) one is free to define $u^a(x)$ as they wish provided $\lim_{x\to\bar{x}} u^a(x) = u^{\bar{a}}$. A natural covariant choice would be to define this through parallel transport, $u^a(x) = g^a{}_{\bar{b}} u^{\bar{b}}$. However, in reality it is more practical in numerical calculations to define u^a such that its components in Boyer-Lindquist coordinates are equal to the components of $u^{\bar{a}}$ in Boyer-Lindquist coordinates [41]. Doing so, the regularization parameters are given by

$$
\begin{aligned}
F_{t[\text{-}1]} &= -\frac{r_0 \dot{r}_0 \operatorname{sgn}\Delta r}{r_0\left(a^2+L^2\right)+2a^2M+r_0^3},\\
F_{r[\text{-}1]} &= \frac{\operatorname{sgn}\Delta r\left(Er_0\left(a^2+r_0^2\right)+2aM(aE-L)\right)}{\left(a^2-2Mr_0+r_0^2\right)\left(r_0\left(a^2+L^2\right)+2a^2M+r_0^3\right)},\\
F_{\theta[\text{-}1]} &= 0, \quad F_{\phi[\text{-}1]} = 0,
\end{aligned} \tag{2.56}
$$

$$F_{t[0]} = \frac{\dot{r}_0}{\pi r_0^2\left(r_0^2+L^2+\frac{2a^2M}{r_0}+a^2\right)^{3/2}\left(2a^2M+a^2r_0+L^2r_0\right)^2}\left(F^{\mathcal{E}}_{t[0]}\mathcal{E}+F^{\mathcal{K}}_{t[0]}\mathcal{K}\right), \tag{2.57}$$

where

$$
\begin{aligned}
F^{\mathcal{E}}_{t[0]} &= -4aLM\left(4a^4M^2+2a^4Mr_0+2a^2L^2Mr_0-a^2Mr_0^3-a^2r_0^4-L^2r_0^4\right)\\
&\quad+E\big(-12a^6M^3-16a^6M^2r_0-7a^6Mr_0^2-a^6r_0^3-28a^4L^2M^2r_0-22a^4L^2Mr_0^2-4a^4L^2r_0^3-6a^4M^2r_0^3\\
&\qquad-5a^4Mr_0^4-a^4r_0^5-15a^2L^4Mr_0^2-5a^2L^4r_0^3-5a^2L^2Mr_0^4-a^2L^2r_0^5-2L^6r_0^3\big),\\
F^{\mathcal{K}}_{t[0]} &= 2aLM\big(2a^4M^2-a^4Mr_0-a^4r_0^2-a^2L^2Mr_0-2a^2L^2r_0^2-2a^2Mr_0^3-2a^2r_0^4-L^4r_0^2-2L^2r_0^4\big)\\
&\quad+E\big(4a^6M^3+4a^6M^2r_0+a^6Mr_0^2+10a^4L^2M^2r_0+5a^4L^2Mr_0^2+2a^4M^2r_0^3+a^4Mr_0^4+4a^2L^4Mr_0^2\\
&\qquad+a^2L^2Mr_0^4-a^2L^2r_0^5-L^4r_0^5\big),
\end{aligned}
$$

$$F_{r[0]} = \frac{\left(F^{\mathcal{E}}_{t[0]}\mathcal{E}+F^{\mathcal{K}}_{t[0]}\mathcal{K}\right)}{\pi r_0^3\left(r_0^2+L^2+\frac{2a^2M}{r_0}+a^2\right)^{3/2}\left(2a^2M+a^2r_0+L^2r_0\right)^2\left(a^2-2Mr_0+r_0^2\right)}, \tag{2.58}$$

where

$$
\begin{aligned}
F^{\mathcal{E}}_{r[0]} = {} & L^2\big(24a^6M^4+28a^6M^3r_0-6a^6M^2r_0^2-11a^6Mr_0^3-2a^6r_0^4+56a^4L^2M^3r_0+24a^4L^2M^2r_0^2-18a^4L^2Mr_0^3 \\
& -6a^4L^2r_0^4+52a^4M^3r_0^3+20a^4M^2r_0^4-11a^4Mr_0^5-3a^4r_0^6+30a^2L^4M^2r_0^2-3a^2L^4Mr_0^3-6a^2L^4r_0^4 \\
& +42a^2L^2M^2r_0^4-5a^2L^2Mr_0^5-6a^2L^2r_0^6+8a^2M^2r_0^6-2a^2Mr_0^7-a^2r_0^8+4L^6Mr_0^3-2L^6r_0^4+6L^4Mr_0^5 \\
& -3L^4r_0^6+2L^2Mr_0^7-L^2r_0^8\big) \\
& -2aELM\big(24a^6M^3+36a^6M^2r_0+18a^6Mr_0^2+3a^6r_0^3+56a^4L^2M^2r_0+48a^4L^2Mr_0^2+10a^4L^2r_0^3+24a^4M^2r_0^3 \\
& +24a^4Mr_0^4+6a^4r_0^5+30a^2L^4Mr_0^2+11a^2L^4r_0^3+22a^2L^2Mr_0^4+9a^2L^2r_0^5+6a^2Mr_0^6+3a^2r_0^7+4L^6r_0^3 \\
& +3L^4r_0^5+3L^2r_0^7\big) \\
& +E^2\left(2a^2M+a^2r_0+r_0^3\right)\big(12a^6M^3+16a^6M^2r_0+7a^6Mr_0^2+a^6r_0^3+28a^4L^2M^2r_0+22a^4L^2Mr_0^2+4a^4L^2r_0^3 \\
& +6a^4M^2r_0^3+5a^4Mr_0^4+a^4r_0^5+15a^2L^4Mr_0^2+5a^2L^4r_0^3+5a^2L^2Mr_0^4+a^2L^2r_0^5+2L^6r_0^3\big), \\
F^{\mathcal{K}}_{r[0]} = {} & -L^2\big(8a^6M^4+12a^6M^3r_0-2a^6Mr_0^3+20a^4L^2M^3r_0+8a^4L^2M^2r_0^2-4a^4L^2Mr_0^3+32a^4M^3r_0^3+12a^4M^2r_0^4 \\
& -6a^4Mr_0^5-a^4r_0^6+8a^2L^4M^2r_0^2-2a^2L^4Mr_0^3+24a^2L^2M^2r_0^4-4a^2L^2Mr_0^5-2a^2L^2r_0^6+8a^2M^2r_0^6 \\
& -2a^2Mr_0^7-a^2r_0^8+2L^4Mr_0^5-L^4r_0^6+2L^2Mr_0^7-L^2r_0^8\big) \\
& +2aELM\big(8a^6M^3+12a^6M^2r_0+6a^6Mr_0^2+a^6r_0^3+20a^4L^2M^2r_0+14a^4L^2Mr_0^2+2a^4L^2r_0^3+16a^4M^2r_0^3 \\
& +16a^4Mr_0^4+4a^4r_0^5+8a^2L^4Mr_0^2+a^2L^4r_0^3+14a^2L^2Mr_0^4+5a^2L^2r_0^5+6a^2Mr_0^6+3a^2r_0^7+L^4r_0^5+3L^2r_0^7\big) \\
& -E^2\left(2a^2M+a^2r_0+r_0^3\right)\big(4a^6M^3+4a^6M^2r_0+a^6Mr_0^2+10a^4L^2M^2r_0+5a^4L^2Mr_0^2+2a^4M^2r_0^3+a^4Mr_0^4 \\
& +4a^2L^4Mr_0^2+a^2L^2Mr_0^4-a^2L^2r_0^5-L^4r_0^5\big),
\end{aligned}
$$

$$
F_{\theta[0]} = 0 \tag{2.59}
$$

$$
F_{\phi[0]} = \frac{L\dot{r}_0\left(F^{\mathcal{E}}_{\phi[0]}\mathcal{E}+F^{\mathcal{K}}_{\phi[0]}\mathcal{K}\right)}{\pi r_0\left(r_0^2+L^2+\frac{2a^2M}{r_0}+a^2\right)^{1/2}\left(2a^2M+a^2r_0+L^2r_0\right)^2}, \tag{2.60}
$$

where

$$
\begin{aligned}
F^{\mathcal{E}}_{\phi[0]} &= 14a^4M^2+11a^4Mr_0+2a^4r_0^2+11a^2L^2Mr_0+4a^2L^2r_0^2+4a^2Mr_0^3+a^2r_0^4+2L^4r_0^2+L^2r_0^4, \\
F^{\mathcal{K}}_{\phi[0]} &= -4a^4M^2-2a^4Mr_0-2a^2L^2Mr_0-4a^2Mr_0^3-a^2r_0^4-L^2r_0^4.
\end{aligned}
$$

As with the Scalar case, $F_{a[2]}$ proves too large to include in paper format and so is available electronically [56], although we provide $F_{r[2]}$ for circular orbits below to allow the reader to get an understanding of the structure of the parameters.

$$
\begin{aligned}
F_{r[2]} = {} & \frac{1}{6\pi M r_0^{11/2}}\sqrt{\frac{2aM^{5/2}r_0^{3/2}+M^2r_0^2(r_0-3M)}{\left(a\sqrt{M}+r_0^{3/2}\right)^2\left[a^2+r_0(r_0-2M)\right]}} \\
& \times\frac{\left(2a\sqrt{M}+\sqrt{r_0}(r_0-3M)\right)^{-2}\left(F^{\mathcal{E}}_{r[2]}\mathcal{E}+F^{\mathcal{K}}_{r[2]}\mathcal{K}\right)}{\left(a^4M+2a^3\sqrt{Mr_0^3}+a^2r_0\left(-2M^2+Mr_0+r_0^2\right)-4aM^{3/2}r_0^{5/2}+Mr_0^4\right)^3},
\end{aligned} \tag{2.61}
$$

where

$$
\begin{aligned}
F^{\mathcal{E}}_{r[2]} = &-\left(r_0^{3/2}+a\sqrt{M}\right)^2\big(-3M^3r_0^{33/2}-96M^4r_0^{31/2}-9a^2M^2r_0^{31/2}+438aM^{7/2}r_0^{15}+495M^5r_0^{29/2}-399a^2M^3r_0^{29/2}\\
&-9a^4Mr_0^{29/2}-2088aM^{9/2}r_0^{14}+388a^3M^{5/2}r_0^{14}-3a^6r_0^{27/2}-588M^6r_0^{27/2}+1080a^2M^4r_0^{27/2}\\
&-228a^4M^2r_0^{27/2}+2058aM^{11/2}r_0^{13}-372a^3M^{7/2}r_0^{13}+54a^5M^{3/2}r_0^{13}+4467a^2M^5r_0^{25/2}+599a^4M^3r_0^{25/2}\\
&-123a^6Mr_0^{25/2}+828aM^{13/2}r_0^{12}-7504a^3M^{9/2}r_0^{12}+644a^5M^{5/2}r_0^{12}-24a^7\sqrt{M}r_0^{12}-6a^8r_0^{23/2}\\
&-11265a^2M^6r_0^{23/2}+1578a^4M^4r_0^{23/2}+609a^6M^2r_0^{23/2}+9576a^3M^{11/2}r_0^{11}-5882a^5M^{7/2}r_0^{11}\\
&-528a^7M^{3/2}r_0^{11}+2358a^2M^7r_0^{21/2}+14389a^4M^5r_0^{21/2}+3516a^6M^3r_0^{21/2}-189a^8Mr_0^{21/2}\\
&+7608a^3M^{13/2}r_0^{10}-2650a^5M^{9/2}r_0^{10}+3296a^7M^{5/2}r_0^{10}-48a^9\sqrt{M}r_0^{10}-39138a^4M^6r_0^{19/2}\\
&-16229a^6M^4r_0^{19/2}-330a^8M^2r_0^{19/2}-2184a^3M^{15/2}r_0^{9}+21172a^5M^{11/2}r_0^{9}-3530a^7M^{7/2}r_0^{9}-750a^9M^{3/2}r_0^{9}\\
&+21603a^4M^7r_0^{17/2}+39699a^6M^5r_0^{17/2}+8911a^8M^3r_0^{17/2}-156a^{10}Mr_0^{17/2}-29982a^5M^{13/2}r_0^{8}\\
&-26366a^7M^{9/2}r_0^{8}+1576a^9M^{5/2}r_0^{8}-5706a^4M^8r_0^{15/2}-16697a^6M^6r_0^{15/2}-16348a^8M^4r_0^{15/2}\\
&-1368a^{10}M^2r_0^{15/2}+13800a^5M^{15/2}r_0^{7}+39338a^7M^{11/2}r_0^{7}+9096a^9M^{7/2}r_0^{7}-264a^{11}M^{3/2}r_0^{7}\\
&+2436a^6M^7r_0^{13/2}+1368a^8M^5r_0^{13/2}+3426a^{10}M^3r_0^{13/2}-3096a^5M^{17/2}r_0^{6}-23546a^7M^{13/2}r_0^{6}\\
&-16754a^9M^{9/2}r_0^{6}-1032a^{11}M^{5/2}r_0^{6}+4404a^6M^8r_0^{11/2}+4464a^8M^6r_0^{11/2}+654a^{10}M^4r_0^{11/2}\\
&-246a^{12}M^2r_0^{11/2}+5880a^7M^{15/2}r_0^{5}+15222a^9M^{11/2}r_0^{5}+2686a^{11}M^{7/2}r_0^{5}-516a^6M^9r_0^{9/2}-9146a^8M^7r_0^{9/2}\\
&-3435a^{10}M^5r_0^{9/2}+54a^{12}M^3r_0^{9/2}+600a^7M^{17/2}r_0^{4}-4018a^9M^{13/2}r_0^{4}-4530a^{11}M^{9/2}r_0^{4}-120a^{13}M^{5/2}r_0^{4}\\
&+1164a^8M^8r_0^{7/2}+6992a^{10}M^6r_0^{7/2}+702a^{12}M^4r_0^{7/2}-1288a^9M^{15/2}r_0^{3}+1158a^{11}M^{11/2}r_0^{3}+540a^{13}M^{7/2}r_0^{3}\\
&-969a^{10}M^7r_0^{5/2}-2336a^{12}M^5r_0^{5/2}-24a^{14}M^3r_0^{5/2}+1030a^{11}M^{13/2}r_0^{2}-116a^{13}M^{9/2}r_0^{2}+354a^{12}M^6r_0^{3/2}\\
&+288a^{14}M^4r_0^{3/2}-364a^{13}M^{11/2}r_0-48a^{14}M^5\sqrt{r_0}+48a^{15}M^{9/2}\big),
\end{aligned}
$$

$$
\begin{aligned}
F^{\mathcal{E}}_{r[2]} = &-r_0^3\left(r_0^{3/2}-3M\sqrt{r_0}+2a\sqrt{M}\right)\big(3M^3r_0^{15}+96M^4r_0^{14}+9a^2M^2r_0^{14}-438aM^{7/2}r_0^{27/2}-255M^5r_0^{13}\\
&+393a^2M^3r_0^{13}+9a^4Mr_0^{13}+1014aM^{9/2}r_0^{25/2}-388a^3M^{5/2}r_0^{25/2}+3a^6r_0^{12}-69a^2M^4r_0^{12}+210a^4M^2r_0^{12}\\
&+252aM^{11/2}r_0^{23/2}-406a^3M^{7/2}r_0^{23/2}-54a^5M^{3/2}r_0^{23/2}-2763a^2M^5r_0^{11}-32a^4M^3r_0^{11}+105a^6Mr_0^{11}\\
&+3594a^3M^{9/2}r_0^{21/2}-738a^5M^{5/2}r_0^{21/2}+954a^2M^6r_0^{10}+763a^4M^4r_0^{10}-204a^6M^2r_0^{10}-2940a^3M^{11/2}r_0^{19/2}\\
&+1610a^5M^{7/2}r_0^{19/2}+474a^7M^{3/2}r_0^{19/2}-3648a^4M^5r_0^{9}-3014a^6M^3r_0^{9}+222a^8Mr_0^{9}-564a^3M^{13/2}r_0^{17/2}\\
&+2656a^5M^{9/2}r_0^{17/2}-2366a^7M^{5/2}r_0^{17/2}+10731a^4M^6r_0^{8}+12274a^6M^4r_0^{8}+963a^8M^2r_0^{8}\\
&-21096a^5M^{11/2}r_0^{15/2}-3328a^7M^{7/2}r_0^{15/2}+996a^9M^{3/2}r_0^{15/2}-1881a^4M^7r_0^{7}-7640a^6M^5r_0^{7}-9303a^8M^3r_0^{7}\\
&+17622a^5M^{13/2}r_0^{13/2}+30698a^7M^{9/2}r_0^{13/2}+1056a^9M^{5/2}r_0^{13/2}-21807a^6M^6r_0^{6}+517a^8M^4r_0^{6}\\
&+2283a^{10}M^2r_0^{6}-1044a^5M^{15/2}r_0^{11/2}-16022a^9M^{7/2}r_0^{11/2}+8817a^6M^7r_0^{5}+30400a^8M^5r_0^{5}+426a^{10}M^3r_0^{5}\\
&-8268a^7M^{13/2}r_0^{9/2}+4904a^9M^{9/2}r_0^{9/2}+2940a^{11}M^{5/2}r_0^{9/2}-174a^6M^8r_0^{4}-10712a^8M^6r_0^{4}-13842a^{10}M^4r_0^{4}\\
&+1428a^7M^{15/2}r_0^{7/2}+12492a^9M^{11/2}r_0^{7/2}-360a^{11}M^{7/2}r_0^{7/2}-1011a^8M^7r_0^{3}+4108a^{10}M^5r_0^{3}+2172a^{12}M^3r_0^{3}\\
&-2062a^9M^{13/2}r_0^{5/2}-5774a^{11}M^{9/2}r_0^{5/2}+1725a^{10}M^6r_0^{2}-480a^{12}M^4r_0^{2}+986a^{11}M^{11/2}r_0^{3/2}\\
&+864a^{13}M^{7/2}r_0^{3/2}-876a^{12}M^5r_0-17222a^7(Mr_0)^{11/2}+144a^{14}M^4-156a^{13}\sqrt{M^9r_0}+24a^7\sqrt{Mr_0^{21}}\big)
\end{aligned}
$$

F. Gravitational l-mode Regularization Parameters

1. Self-force regularization

The singular part of the self force on a gravitational particle is given by

$$F^a = k^{abcd}\bar{h}^{(\mathrm{S})}_{bc;d}, \tag{2.62}$$

where

$$k^{abcd} \equiv \frac{1}{2}g^{ad}u^bu^c - g^{ab}u^cu^d - \frac{1}{2}u^au^bu^cu^d + \frac{1}{4}u^ag^{bc}u^d + \frac{1}{4}g^{ad}g^{bc}, \tag{2.63}$$

and $\bar{h}^{(\mathrm{S})}_{bc}$ is the trace-reversed singular field. Note that, as in the electromagnetic case, an ambiguity arises here due to the presence of terms involving the four-velocity at x. One is free to arbitrarily choose how to define this provided $\lim_{x\to\bar{x}} u^a = u^{\bar{a}}$. Following Barack and Sago [41], we choose to take the Boyer-Lindquist components of the four velocity at x to be exactly those at $\bar{x}$. The regularisation parameters in the gravitational case are given by,

$$F^t{}_{[\text{-1}]} = \frac{r_0 \dot{r}_0 \mathrm{sgn}\Delta \mathrm{r}}{r_0\left(a^2+L^2\right)+2a^2M+r_0^3}, \tag{2.64}$$

$$F^r{}_{[\text{-1}]} = -\frac{\mathrm{sgn}\Delta\mathrm{r}\left(Er_0\left(a^2+r_0^2\right)+2aM(aE-L)\right)}{\left(a^2-2Mr_0+r_0^2\right)\left(r_0\left(a^2+L^2\right)+2a^2M+r_0^3\right)}, \tag{2.65}$$

$$F^\theta{}_{[\text{-1}]} = 0, \qquad F^\phi{}_{[\text{-1}]} = 0, \tag{2.66}$$

$$F^t{}_{[0]} = \frac{\dot{r}_0\left(F^t_{\mathcal{E}[0]}\mathcal{E}+F^t_{\mathcal{K}[0]}\mathcal{K}\right)}{\pi r_0^2\left(r_0^2+L^2+\frac{2a^2M}{r_0}+a^2\right)^{3/2}\left(2a^2M+a^2r_0+L^2r_0\right)^2}, \tag{2.67}$$

where

$$\begin{aligned}
F^t_{\mathcal{E}[0]} = {}& 4aLM\left(4a^4M^2+2a^4Mr_0+2a^2L^2Mr_0-a^2Mr_0^3-a^2r_0^4-L^2r_0^4\right)\\
&+E\big(12a^6M^3+16a^6M^2r_0+7a^6Mr_0^2+a^6r_0^3+28a^4L^2M^2r_0+22a^4L^2Mr_0^2+4a^4L^2r_0^3+6a^4M^2r_0^3+5a^4Mr_0^4\\
&\quad+a^4r_0^5+15a^2L^4Mr_0^2+5a^2L^4r_0^3+5a^2L^2Mr_0^4+a^2L^2r_0^5+2L^6r_0^3\big),\\
F^t_{\mathcal{K}[0]} = {}& -2aLM\big(2a^4M^2-a^4Mr_0-a^4r_0^2-a^2L^2Mr_0-2a^2L^2r_0^2-2a^2Mr_0^3-2a^2r_0^4-L^4r_0^2-2L^2r_0^4\big)\\
&+E\big(-4a^6M^3-4a^6M^2r_0-a^6Mr_0^2-10a^4L^2M^2r_0-5a^4L^2Mr_0^2-2a^4M^2r_0^3-a^4Mr_0^4-4a^2L^4Mr_0^2\\
&\quad-a^2L^2Mr_0^4+a^2L^2r_0^5+L^4r_0^5\big),
\end{aligned}$$

$$F^r{}_{[0]} = \frac{\left(F^r_{\mathcal{E}[0]}\mathcal{E}+F^r_{\mathcal{K}[0]}\mathcal{K}\right)}{\pi r_0^6\left(r_0^2+L^2+\frac{2a^2M}{r_0}+a^2\right)^{3/2}\left(2a^2M+a^2r_0+L^2r_0\right)^2\left(a^2-2Mr_0+r_0^2\right)}, \tag{2.68}$$

where

$$\begin{aligned}
F^r_{\mathcal{E}[0]} = {}& +L^2r_0^3\big(-24a^6M^4-28a^6M^3r_0+6a^6M^2r_0^2+11a^6Mr_0^3+2a^6r_0^4-56a^4L^2M^3r_0-24a^4L^2M^2r_0^2\\
&\quad+18a^4L^2Mr_0^3+6a^4L^2r_0^4-52a^4M^3r_0^3-20a^4M^2r_0^4+11a^4Mr_0^5+3a^4r_0^6-30a^2L^4M^2r_0^2+3a^2L^4Mr_0^3\\
&\quad+6a^2L^4r_0^4-42a^2L^2M^2r_0^4+5a^2L^2Mr_0^5+6a^2L^2r_0^6-8a^2M^2r_0^6+2a^2Mr_0^7+a^2r_0^8-4L^6Mr_0^3+2L^6r_0^4\\
&\quad-6L^4Mr_0^5+3L^4r_0^6-2L^2Mr_0^7+L^2r_0^8\big)\\
&+2aELMr_0^3\big(24a^6M^3+36a^6M^2r_0+18a^6Mr_0^2+3a^6r_0^3+56a^4L^2M^2r_0+48a^4L^2Mr_0^2+10a^4L^2r_0^3\\
&\quad+24a^4M^2r_0^3+24a^4Mr_0^4+6a^4r_0^5+30a^2L^4Mr_0^2+11a^2L^4r_0^3+22a^2L^2Mr_0^4+9a^2L^2r_0^5+6a^2Mr_0^6\\
&\quad+3a^2r_0^7+4L^6r_0^3+3L^4r_0^5+3L^2r_0^7\big)\\
&-E^2r_0^3\left(2a^2M+a^2r_0+r_0^3\right)\big(12a^6M^3+16a^6M^2r_0+7a^6Mr_0^2+a^6r_0^3+28a^4L^2M^2r_0+22a^4L^2Mr_0^2\\
&\quad+4a^4L^2r_0^3+6a^4M^2r_0^3+5a^4Mr_0^4+a^4r_0^5+15a^2L^4Mr_0^2+5a^2L^4r_0^3+5a^2L^2Mr_0^4+a^2L^2r_0^5+2L^6r_0^3\big),\\
F^r_{\mathcal{K}[0]} = {}& -L^2r_0^3\big(-8a^6M^4-12a^6M^3r_0+2a^6Mr_0^3-20a^4L^2M^3r_0-8a^4L^2M^2r_0^2+4a^4L^2Mr_0^3-32a^4M^3r_0^3\\
&\quad-12a^4M^2r_0^4+6a^4Mr_0^5+a^4r_0^6-8a^2L^4M^2r_0^2+2a^2L^4Mr_0^3-24a^2L^2M^2r_0^4+4a^2L^2Mr_0^5+2a^2L^2r_0^6\\
&\quad-8a^2M^2r_0^6+2a^2Mr_0^7+a^2r_0^8-2L^4Mr_0^5+L^4r_0^6-2L^2Mr_0^7+L^2r_0^8\big)\\
&-2aELMr_0^3\big(8a^6M^3+12a^6M^2r_0+6a^6Mr_0^2+a^6r_0^3+20a^4L^2M^2r_0+14a^4L^2Mr_0^2+2a^4L^2r_0^3+16a^4M^2r_0^3\\
&\quad+16a^4Mr_0^4+4a^4r_0^5+8a^2L^4Mr_0^2+a^2L^4r_0^3+14a^2L^2Mr_0^4+5a^2L^2r_0^5+6a^2Mr_0^6+3a^2r_0^7+L^4r_0^5+3L^2r_0^7\big)\\
&-E^2r_0^3\left(2a^2M+a^2r_0+r_0^3\right)\big(-4a^6M^3-4a^6M^2r_0-a^6Mr_0^2-10a^4L^2M^2r_0-5a^4L^2Mr_0^2-2a^4M^2r_0^3\\
&\quad-a^4Mr_0^4-4a^2L^4Mr_0^2-a^2L^2Mr_0^4+a^2L^2r_0^5+L^4r_0^5\big),
\end{aligned}$$

$$F^\theta{}_{[0]} = 0 \tag{2.69}$$

$$F^{\phi}{}_{[0]} = \frac{\left(F^{\phi}_{\mathcal{E}\,[0]}\mathcal{E} + F^{\phi}_{\mathcal{K}\,[0]}\mathcal{K}\right)}{\pi r_0^6\left(r_0^2 + L^2 + \frac{2a^2M}{r_0} + a^2\right)^{3/2}\left(2a^2M + a^2r_0 + L^2r_0\right)^2\left(a^2 - 2Mr_0 + r_0^2\right)}, \tag{2.70}$$

where

$$F^{\phi}_{\mathcal{E}\,[0]} = -14a^4M^2 - 11a^4Mr_0 - 2a^4r_0^2 - 11a^2L^2Mr_0 - 4a^2L^2r_0^2 - 4a^2Mr_0^3 - a^2r_0^4 - 2L^4r_0^2 - L^2r_0^4,$$
$$F^{\phi}_{\mathcal{K}\,[0]} = 4a^4M^2 + 2a^4Mr_0 + 2a^2L^2Mr_0 + 4a^2Mr_0^3 + a^2r_0^4 + L^2r_0^4.$$

As with the scalar and electromagnetic cases, $F^a_{[2]}$ is too large for paper format and so is available electronically [56]. Instead, we give here $F^r_{[2]}$ for circular orbits,

$$\begin{aligned} F^{r}{}_{[2]} = {} & \frac{1}{\pi r_0^{11/2}}\sqrt{\frac{2aM^{5/2}r_0^{3/2} + M^2r_0^2(r_0 - 3M)}{\left(a\sqrt{M} + r_0^{3/2}\right)^2\left[a^2 + r_0(r_0 - 2M)\right]}}\left[2a\sqrt{M} + \sqrt{r_0}(r_0 - 3M)\right]^{-2} \\ & \times \frac{\left[a^2 + r_0(r_0 - 2M)\right]^{-1}\left(F^r_{\mathcal{E}\,[2]}\mathcal{E} + F^r_{\mathcal{K}\,[2]}\mathcal{K}\right)}{\left(a^4M + 2a^3\sqrt{Mr_0^3} + a^2r_0\left(-2M^2 + Mr_0 + r_0^2\right) - 4aM^{3/2}r_0^{5/2} + Mr_0^4\right)^2}, \end{aligned}$$

where

$$\begin{aligned} F^r_{\mathcal{E}\,[2]} = {} & -\left(a\sqrt{M} + r_0^{3/2}\right)^2\left(a^2 - 2Mr_0 + r_0^2\right)\big(32a^{11}M^{5/2} - 48a^{10}M^3\sqrt{r_0} + 144a^{10}M^2r_0^{3/2} + 478a^9M^{3/2}r_0^3 \\ & - 104a^9M^{5/2}r_0^2 - 192a^9M^{7/2}r_0 + 288a^8M^4r_0^{3/2} - 992a^8M^3r_0^{5/2} - 473a^8M^2r_0^{7/2} + 779a^8Mr_0^{9/2} \\ & - 292a^7M^{3/2}r_0^5 - 2496a^7M^{5/2}r_0^4 + 800a^7M^{7/2}r_0^3 + 384a^7M^{9/2}r_0^2 + 540a^7\sqrt{M}r_0^6 - 576a^6M^5r_0^{5/2} \\ & + 2240a^6M^4r_0^{7/2} + 2489a^6M^3r_0^{9/2} - 4623a^6M^2r_0^{11/2} + 227a^6Mr_0^{13/2} + 127a^6r_0^{15/2} - 1846a^5M^{3/2}r_0^7 \\ & + 3574a^5M^{5/2}r_0^6 + 4018a^5M^{7/2}r_0^5 - 1952a^5M^{9/2}r_0^4 - 256a^5M^{11/2}r_0^3 + 386a^5\sqrt{M}r_0^8 + 384a^4M^6r_0^{7/2} \\ & - 1664a^4M^5r_0^{9/2} - 3950a^4M^4r_0^{11/2} + 6038a^4M^3r_0^{13/2} - 2275a^4M^2r_0^{15/2} - 462a^4Mr_0^{17/2} + 105a^4r_0^{19/2} \\ & - 14a^3M^{3/2}r_0^9 + 4444a^3M^{5/2}r_0^8 - 6514a^3M^{7/2}r_0^7 - 1876a^3M^{9/2}r_0^6 + 1536a^3M^{11/2}r_0^5 - 144a^3\sqrt{M}r_0^{10} \\ & + 1728a^2M^5r_0^{13/2} + 674a^2M^4r_0^{15/2} + 588a^2M^3r_0^{17/2} - 1312a^2M^2r_0^{19/2} + 348a^2Mr_0^{21/2} - 22a^2r_0^{23/2} \\ & - 148aM^{3/2}r_0^{11} + 884aM^{5/2}r_0^{10} - 1370aM^{7/2}r_0^9 + 384aM^{9/2}r_0^8 - 192M^4r_0^{19/2} + 287M^3r_0^{21/2} \\ & - 131M^2r_0^{23/2} + 18Mr_0^{25/2}\big), \\ F^r_{\mathcal{K}\,[2]} = {} & r_0^3\left(2a\sqrt{M} - 3M\sqrt{r_0} + r_0^{3/2}\right)\big(144a^{12}M^2 - 96a^{11}\sqrt{M^5r_0} + 576a^{11}(Mr_0)^{3/2} - 921a^{10}M^3r_0 + 81a^{10}M^2r_0^2 \\ & + 864a^{10}Mr_0^3 + 862a^9M^{3/2}r_0^{7/2} + 582a^9M^{7/2}r_0^{3/2} + 576a^9\sqrt{Mr_0^9} - 4010a^9(Mr_0)^{5/2} + 2052a^8M^4r_0^2 \\ & + 313a^8M^3r_0^3 - 5342a^8M^2r_0^4 + 1351a^8Mr_0^5 + 144a^8r_0^6 - 2308a^7M^{3/2}r_0^{11/2} - 3816a^7M^{5/2}r_0^{9/2} \\ & - 1176a^7M^{9/2}r_0^{5/2} + 920a^7\sqrt{Mr_0^{13}} + 9524a^7(Mr_0)^{7/2} - 1764a^6M^5r_0^3 - 2260a^6M^4r_0^4 + 12726a^6M^3r_0^5 \\ & - 6148a^6M^2r_0^6 + 60a^6Mr_0^7 + 254a^6r_0^8 - 2580a^5M^{3/2}r_0^{15/2} + 4204a^5M^{5/2}r_0^{13/2} + 3504a^5M^{7/2}r_0^{11/2} \\ & + 792a^5M^{11/2}r_0^{7/2} + 204a^5\sqrt{Mr_0^{17}} - 8384a^5(Mr_0)^{9/2} + 384a^4M^6r_0^4 + 2620a^4M^5r_0^5 - 11878a^4M^4r_0^6 \\ & + 8425a^4M^3r_0^7 - 1656a^4M^2r_0^8 - 111a^4Mr_0^9 + 88a^4r_0^{10} - 242a^3M^{3/2}r_0^{19/2} + 2928a^3M^{5/2}r_0^{17/2} \\ & - 4866a^3M^{7/2}r_0^{15/2} + 1372a^3M^{9/2}r_0^{13/2} - 144a^3\sqrt{Mr_0^{21}} + 1536a^3(Mr_0)^{11/2} + 1728a^2M^5r_0^7 - 1614a^2M^4r_0^8 \\ & + 1373a^2M^3r_0^9 - 1075a^2M^2r_0^{10} + 334a^2Mr_0^{11} - 22a^2r_0^{12} - 148aM^{3/2}r_0^{23/2} + 718aM^{5/2}r_0^{21/2} \\ & - 1040aM^{7/2}r_0^{19/2} + 384aM^{9/2}r_0^{17/2} - 192M^4r_0^{10} + 280M^3r_0^{11} - 128M^2r_0^{12} + 18Mr_0^{13}\big). \end{aligned}$$

2. huu regularization

The quantity

$$H^{(R)} = \frac{1}{2}h^{(R)}_{ab}u^a u^b \tag{2.71}$$

was first proposed by Detweiler [57] as a tool for constructing gauge invariant measurements from self-force calculations. It has since proven invaluable in extracting gauge invariant results from gauge dependent self-force calculations [39, 58].

Much the same as with self-force calculations, the calculation of $H^{(R)}$ requires the subtraction of the appropriate singular piece, $H^{(S)} = \frac{1}{2}h^{(S)}_{ab}u^a u^b$ from the full retarded field. In this section, we give this subtraction in the form of mode-sum regularization parameters. In doing so, we keep with our convention that the term proportional to $l+\frac{1}{2}$ is denoted by $H_{[-1]}$ ($=0$ in this case), the constant term is denoted by $H_{[0]}$, and so on.

Note that, as in the self-force case, an ambiguity arises here due to the presence of terms involving the four-velocity at x. One is free to arbitrarily choose how to define this provided $\lim_{x\to\bar{x}} u^a = u^{\bar{a}}$. As before, we choose this in such a way that the Boyer-Lindquist components of the four velocity at x are exactly those at $\bar{x}$. The regularisation parameters are then given by,

$$H_{[0]} = \frac{2\mathcal{K}}{\pi\left(r_0^2 + L^2 + \frac{2a^2M}{r_0} + a^2\right)^{1/2}}, \tag{2.72}$$

$$H_{[1]} = 0, \tag{2.73}$$

$$H_{[2]} = \frac{\left(H^{\mathcal{E}}_{[2]}\mathcal{E} + H^{\mathcal{K}}_{[2]}\mathcal{K}\right)}{3\pi r_0^7\left(r_0^2 + L^2 + \frac{2a^2M}{r_0} + a^2\right)^{3/2}\left(2a^2M + a^2r_0 + L^2r_0\right)^3}, \tag{2.74}$$

where

$$\begin{aligned}
H^{\mathcal{E}}_{[2]} = \big(&12Mr_0^5a^{12} + 92M^2r_0^4a^{12} + 264M^3r_0^3a^{12} + 336M^4r_0^2a^{12} + 160M^5r_0a^{12} - 24r_0^8a^{10} - 240Mr_0^7a^{10} \\
&- 1104L^2M^6a^{10} - 1104M^2r_0^6a^{10} - 2880M^3r_0^5a^{10} + 48L^2Mr_0^5a^{10} - 4272M^4r_0^4a^{10} + 230L^2M^2r_0^4a^{10} \\
&- 3264M^5r_0^3a^{10} + 96L^2M^3r_0^3a^{10} - 960M^6r_0^2a^{10} - 1116L^2M^4r_0^2a^{10} - 2096L^2M^5r_0a^{10} - 48r_0^{10}a^8 \\
&- 420Mr_0^9a^8 - 120L^2r_0^8a^8 - 1556M^2r_0^8a^8 - 2872M^3r_0^7a^8 - 882L^2Mr_0^7a^8 - 2448M^4r_0^6a^8 - 2781L^2M^2r_0^6a^8 \\
&- 672M^5r_0^5a^8 - 4770L^2M^3r_0^5a^8 + 72L^4Mr_0^5a^8 - 4272L^2M^4r_0^4a^8 + 90L^4M^2r_0^4a^8 - 1440L^2M^5r_0^3a^8 \\
&- 1044L^4M^3r_0^3a^8 - 2928L^4M^4r_0^2a^8 - 2112L^4M^5r_0a^8 - 24r_0^{12}a^6 - 168Mr_0^{11}a^6 - 195L^2r_0^{10}a^6 - 456M^2r_0^{10}a^6 \\
&- 480M^3r_0^9a^6 - 1086L^2Mr_0^9a^6 - 240L^4r_0^8a^6 - 96M^4r_0^8a^6 - 2119L^2M^2r_0^8a^6 - 1528L^2M^3r_0^7a^6 \\
&- 1098L^4Mr_0^7a^6 - 84L^2M^4r_0^6a^6 - 1578L^4M^2r_0^6a^6 - 696L^4M^3r_0^5a^6 + 48L^6Mr_0^5a^6 + 84L^4M^4r_0^4a^6 \\
&- 190L^6M^2r_0^4a^6 - 1320L^6M^3r_0^3a^6 - 1476L^6M^4r_0^2a^6 - 75L^2r_0^{12}a^4 - 246L^2Mr_0^{11}a^4 - 297L^4r_0^{10}a^4 \\
&- 84L^2M^2r_0^{10}a^4 + 216L^2M^3r_0^9a^4 - 690L^4Mr_0^9a^4 - 240L^6r_0^8a^4 + 529L^4M^2r_0^8a^4 + 1374L^4M^3r_0^7a^4 \\
&- 402L^6Mr_0^7a^4 + 771L^6M^2r_0^6a^4 + 1194L^6M^3r_0^5a^4 + 12L^8Mr_0^5a^4 - 190L^8M^2r_0^4a^4 - 444L^8M^3r_0^3a^4 \\
&- 78L^4r_0^{12}a^2 + 36L^4Mr_0^{11}a^2 - 201L^6r_0^{10}a^2 + 384L^4M^2r_0^{10}a^2 + 198L^6Mr_0^9a^2 - 120L^8r_0^8a^2 + 1092L^6M^2r_0^8a^2 \\
&+ 162L^8Mr_0^7a^2 + 672L^8M^2r_0^6a^2 - 48L^{10}M^2r_0^4a^2 - 27L^6r_0^{12} + 114L^6Mr_0^{11} - 51L^8r_0^{10} + 222L^8Mr_0^9 \\
&- 24L^{10}r_0^8 + 108L^{10}Mr_0^7\big) \\
+ 4aELM\big(&-71a^4r_0^{11} - 75L^4r_0^{11} - 146a^2L^2r_0^{11} - 262a^4Mr_0^{10} - 270a^2L^2Mr_0^{10} - 147a^6r_0^9 - 159L^6r_0^9 \\
&- 465a^2L^4r_0^9 - 453a^4L^2r_0^9 - 240a^4M^2r_0^9 - 811a^6Mr_0^8 - 873a^2L^4Mr_0^8 - 1684a^4L^2Mr_0^8 - 76a^8r_0^7 - 78L^8r_0^7 \\
&- 310a^2L^6r_0^7 - 462a^4L^4r_0^7 - 306a^6L^2r_0^7 - 1490a^6M^2r_0^7 - 1542a^4L^2M^2r_0^7 - 912a^6M^3r_0^6 - 522a^8Mr_0^6 \\
&- 549a^2L^6Mr_0^6 - 1620a^4L^4Mr_0^6 - 1593a^6L^2Mr_0^6 - 1323a^8M^2r_0^5 - 1368a^4L^4M^2r_0^5 - 2691a^6L^2M^2r_0^5 \\
&- 1460a^8M^3r_0^4 - 1458a^6L^2M^3r_0^4 + 37a^{10}Mr_0^4 + 24a^2L^8Mr_0^4 + 109a^4L^6Mr_0^4 + 183a^6L^4Mr_0^4 \\
&+ 135a^8L^2Mr_0^4 - 588a^8M^4r_0^3 + 291a^{10}M^2r_0^3 + 222a^4L^6M^2r_0^3 + 735a^6L^4M^2r_0^3 + 804a^8L^2M^2r_0^3 \\
&+ 858a^{10}M^3r_0^2 + 738a^6L^4M^3r_0^2 + 1596a^8L^2M^3r_0^2 + 1124a^{10}M^4r_0 + 1056a^8L^2M^4r_0 + 552a^{10}M^5\big) \\
- 3E^2\big(&368M^6a^{12} + 4Mr_0^5a^{12} + 55M^2r_0^4a^{12} + 280M^3r_0^3a^{12} + 680M^4r_0^2a^{12} + 800M^5r_0a^{12} - 9r_0^8a^{10} \\
&- 124Mr_0^7a^{10} - 604M^2r_0^6a^{10} - 1368M^3r_0^5a^{10} + 12L^2Mr_0^5a^{10} - 1472M^4r_0^4a^{10} + 160L^2M^2r_0^4a^{10} \\
&- 608M^5r_0^3a^{10} + 672L^2M^3r_0^3a^{10} + 1152L^2M^4r_0^2a^{10} + 704L^2M^5r_0a^{10} - 18r_0^{10}a^8 - 224Mr_0^9a^8 - 35L^2r_0^8a^8 \\
&- 901M^2r_0^8a^8 - 1492M^3r_0^7a^8 - 438L^2Mr_0^7a^8 - 884M^4r_0^6a^8 - 1685L^2M^2r_0^6a^8 - 2604L^2M^3r_0^5a^8 \\
&+ 12L^4Mr_0^5a^8 - 1412L^2M^4r_0^4a^8 + 171L^4M^2r_0^4a^8 + 540L^4M^3r_0^3a^8 + 492L^4M^4r_0^2a^8 - 9r_0^{12}a^6 - 96Mr_0^{11}a^6
\end{aligned}$$

$$- 55L^2r_0^{10}a^6 - 278M^2r_0^{10}a^6 - 244M^3r_0^9a^6 - 596L^2Mr_0^9a^6 - 51L^4r_0^8a^6 - 1679L^2M^2r_0^8a^6 - 1414L^2M^3r_0^7a^6 - 572L^4Mr_0^7a^6 - 1567L^4M^2r_0^6a^6 - 1254L^4M^3r_0^5a^6 + 4L^6Mr_0^5a^6 + 82L^6M^2r_0^4a^6 + 148L^6M^3r_0^3a^6 - 20L^2r_0^{12}a^4 - 176L^2Mr_0^{11}a^4 - 56L^4r_0^{10}a^4 - 266L^2M^2r_0^{10}a^4 - 512L^4Mr_0^9a^4 - 33L^6r_0^8a^4 - 774L^4M^2r_0^8a^4 - 326L^6Mr_0^7a^4 - 486L^6M^2r_0^6a^4 + 16L^8M^2r_0^4a^4 - 13L^4r_0^{12}a^2 - 80L^4Mr_0^{11}a^2 - 19L^6r_0^{10}a^2 - 140L^6Mr_0^9a^2 - 8L^8r_0^8a^2 - 68L^8Mr_0^7a^2 - 2L^6r_0^{12}\big),$$

$$\begin{aligned}
H^{\mathcal{K}}_{[2]} = r_0^3\big(&24r_0^5a^{10} + 180Mr_0^4a^{10} + 512M^2r_0^3a^{10} + 656M^3r_0^2a^{10} + 320M^4r_0a^{10} + 93r_0^7a^8 + 624Mr_0^6a^8 - 912L^2M^5a^8 \\
&+ 120L^2r_0^5a^8 + 1448M^2r_0^5a^8 + 976M^3r_0^4a^8 + 636L^2Mr_0^4a^8 - 912M^4r_0^3a^8 + 891L^2M^2r_0^3a^8 - 1152M^5r_0^2a^8 \\
&- 434L^2M^3r_0^2a^8 - 1720L^2M^4r_0a^8 + 69r_0^9a^6 + 354Mr_0^8a^6 + 375L^2r_0^7a^6 + 492M^2r_0^7a^6 - 168M^3r_0^6a^6 \\
&+ 1620L^2Mr_0^6a^6 + 240L^4r_0^5a^6 - 576M^4r_0^5a^6 + 1207L^2M^2r_0^5a^6 - 2720L^2M^3r_0^4a^6 + 732L^4Mr_0^4a^6 \\
&- 3372L^2M^4r_0^3a^6 - 450L^4M^2r_0^3a^6 - 2896L^4M^3r_0^2a^6 - 2052L^4M^4r_0a^6 + 210L^2r_0^9a^4 + 534L^2Mr_0^8a^4 \\
&+ 567L^4r_0^7a^4 - 384L^2M^2r_0^7a^4 - 1224L^2M^3r_0^6a^4 + 1074L^4Mr_0^6a^4 + 240L^6r_0^5a^4 - 2017L^4M^2r_0^5a^4 \\
&- 3726L^4M^3r_0^4a^4 + 180L^6Mr_0^4a^4 - 1525L^6M^2r_0^3a^4 - 1806L^6M^3r_0^2a^4 + 213L^4r_0^9a^2 - 18L^4Mr_0^8a^2 \\
&+ 381L^6r_0^7a^2 - 888L^4M^2r_0^7a^2 - 216L^6Mr_0^6a^2 + 120L^8r_0^5a^2 - 1776L^6M^2r_0^5a^2 - 192L^8Mr_0^4a^2 \\
&- 696L^8M^2r_0^3a^2 + 72L^6r_0^9 - 198L^6Mr_0^8 + 96L^8r_0^7 - 294L^8Mr_0^6 + 24L^{10}r_0^5 - 96L^{10}Mr_0^4\big) \\
+ 4ar_0^3ELM\big(&456M^4a^8 + 45r_0^4a^8 + 322Mr_0^3a^8 + 862M^2r_0^2a^8 + 1024M^3r_0a^8 + 116r_0^6a^6 + 636Mr_0^5a^6 \\
&+ 183L^2r_0^4a^6 + 1162M^2r_0^4a^6 + 708M^3r_0^3a^6 + 992L^2Mr_0^3a^6 + 1765L^2M^2r_0^2a^6 + 1026L^2M^3r_0a^6 + 71r_0^8a^4 \\
&+ 262Mr_0^7a^4 + 358L^2r_0^6a^4 + 240M^2r_0^6a^4 + 1326L^2Mr_0^5a^4 + 279L^4r_0^4a^4 + 1206L^2M^2r_0^4a^4 + 1018L^4Mr_0^3a^4 \\
&+ 903L^4M^2r_0^2a^4 + 146L^2r_0^8a^2 + 270L^2Mr_0^7a^2 + 368L^4r_0^6a^2 + 690L^4Mr_0^5a^2 + 189L^6r_0^4a^2 + 348L^6Mr_0^3a^2 \\
&+ 75L^4r_0^8 + 126L^6r_0^6 + 48L^8r_0^4\big) \\
- 3r_0^3E^2\big(&304M^5a^{10} + 8r_0^5a^{10} + 90Mr_0^4a^{10} + 386M^2r_0^3a^{10} + 796M^3r_0^2a^{10} + 792M^4r_0a^{10} + 32r_0^7a^8 \\
&+ 308Mr_0^6a^8 + 32L^2r_0^5a^8 + 1068M^2r_0^5a^8 + 1600M^3r_0^4a^8 + 304L^2Mr_0^4a^8 + 880M^4r_0^3a^8 + 1003L^2M^2r_0^3a^8 \\
&+ 1388L^2M^3r_0^2a^8 + 684L^2M^4r_0a^8 + 24r_0^9a^6 + 186Mr_0^8a^6 + 113L^2r_0^7a^6 + 458M^2r_0^7a^6 + 364M^3r_0^6a^6 \\
&+ 854L^2Mr_0^6a^6 + 48L^4r_0^5a^6 + 2017L^2M^2r_0^5a^6 + 1522L^2M^3r_0^4a^6 + 370L^4Mr_0^4a^6 + 849L^4M^2r_0^3a^6 \\
&+ 602L^4M^3r_0^2a^6 + 65L^2r_0^9a^4 + 356L^2Mr_0^8a^4 + 146L^4r_0^7a^4 + 446L^2M^2r_0^7a^4 + 776L^4Mr_0^6a^4 + 32L^6r_0^5a^4 \\
&+ 945L^4M^2r_0^5a^4 + 188L^6Mr_0^4a^4 + 232L^6M^2r_0^3a^4 + 58L^4r_0^9a^2 + 170L^4Mr_0^8a^2 + 81L^6r_0^7a^2 + 230L^6Mr_0^6a^2 \\
&+ 8L^8r_0^5a^2 + 32L^8Mr_0^4a^2 + 17L^6r_0^9 + 16L^8r_0^7\big).
\end{aligned}$$

G. Example

To illustrate the effectiveness of the higher-order regularization parameters, we consider as an example the case of a scalar charge on an eccentric geodesic orbit with $E = 0.955492$ and $L = 3.59656$ in a Kerr spacetime with $a = 0.5M$. The self-force in this case was computed in Ref. [52] using a frequency domain calculation of the retarded field in combination with the first two regularization parameters. Figure 1 shows the effect of using higher order regularization parameters on this calculation. As expected, the numerical l-modes computed in Ref. [45] asymptotically fall off as l^{-2}. Our $F^l_{r[2]}$ regularization parameter analytically gives the coefficient of this leading order in $1/l$ behaviour. After subtracting this leading order behaviour from the numerical modes, we find a remainder which falls off as l^{-4}, as expected, with the coefficient given analytically by our $F^l_{r[4]}$ regularization parameter. Upon subtraction of $F^l_{r[4]}$ the remainder falls off as l^{-6}, as anticipated.

III. EFFECTIVE SOURCE AND m-MODE REGULARIZATION

A. Effective Source Approach to the Self-force

The effective source approach – independently proposed by Barack and Golbourn [15] and by Vega and Detweiler [16] – relies on knowledge of the singular field to derive an equation for a regularized field that gives the self-force without any need for post-processed regularization. If the singular field is known exactly, then the regularized field is totally regular and is a solution of the homogeneous wave equation. In reality, exact expressions for the singular

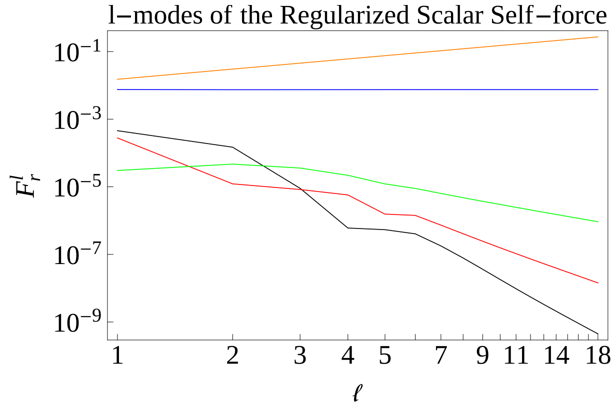

FIG. 1. Regularization of the l-modes of the radial component of the scalar self-force in Kerr spacetime for the case of a particle in an eccentric, equatorial geodesic orbit with $E = 0.955492$ and $L = 3.59656$. The background Kerr black hole has spin $a = 0.5M$. In decreasing slope the lines represent the unregularized self-force (yellow), and the self-force regularized by subtracting from it in turn the cumulative sum of $F^l_{r[-1]}$ (blue), $F^l_{r[0]}$ (green), $F^l_{r[2]}$ (red) and $F^l_{r[4]}$ (black).

field can only be obtained for very simple spacetimes. More generally, the best one can do is an approximation such as the one derived in this paper. Splitting the retarded field into approximate singular and regularized parts,

$$\varphi^A_{(\mathrm{ret})} = \tilde{\varphi}^A_{(\mathrm{S})} + \tilde{\varphi}^A_{(\mathrm{R})}, \tag{3.1}$$

substituting into the wave equation, Eq. (2.1) of Paper I, and rearranging, we obtain an equation for the regularized field,

$$\mathcal{D}^A{}_B \tilde{\varphi}^B_{(\mathrm{R})} = S^A_{\mathrm{eff}}, \tag{3.2}$$

with an *effective source*,

$$S^A_{\mathrm{eff}} = -\mathcal{D}^A{}_B \tilde{\varphi}^B_{(\mathrm{S})} - 4\pi \mathcal{Q} \int u^A \delta_4\left(x, z(\tau')\right) d\tau'. \tag{3.3}$$

For sufficiently good approximations to the singular field, $\tilde{\varphi}^A_{(\mathrm{R})}$ and S^A are finite everywhere, in particular on the world-line. As a result, one never encounters problematic singularities or δ-functions, making the approach particularly suitable for use in time domain numerical simulations. A detailed review of this approach can be found in [59, 60].

One disadvantage of the effective source approach stems from the fact that the source must be evaluated in an extended region around the world-line. Since the source is derived from a complicated expansion approximating the singular field, its evaluation can dominate the run time of a numerical code. This problem is exasperated as increasingly good approximations to the singular field — using increasingly high order series expansions — are used, placing a practical upper limit on the order of singular field approximation which may be used in effective source calculations. Existing calculations [61–63] settled on what appears to be a “sweet spot”, using an approximation accurate to $\mathcal{O}(\epsilon^2)$.

Despite it being possible to compute higher order effective sources from our singular field approximation this practical consideration may appear to rule out the usefulness of high-order expansions of the singular field in effective source calculations, particularly in the case of the Kerr spacetime where even an order $\mathcal{O}(\epsilon^2)$ approximation to the singular field is quite unwieldy. However, it turns out that high-order expansions can, in fact, be put to good use in effective source calculations. In this section, we show how this may be achieved in the case of the m-mode approach to effective source calculations. In this approach, one first performs a decomposition into m-modes

$$\Phi^{(m)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi e^{-im\phi} d\phi. \tag{3.4}$$

and independently evolves the m-decomposed form of the wave equation for each m-mode. These equations have an m dependent effective source which is derived from the particular choice of approximation to the singular field. The full field is then given as a sum of these individual modes:

$$\Phi = \sum_{m=-\infty}^{\infty} \Phi^{(m)} e^{im\phi_0}. \tag{3.5}$$

For an approximation accurate to $\mathcal{O}(\epsilon^n)$, the numerical solutions for the field fall off as $m^{-(n+2)}$ for m even and as $m^{-(n+3)}$ for m odd. Obviously, only finitely many m-modes (typically $\sim 10-20$) can ever be computed numerically; with the error from truncating the sum at a finite m putting an upper limit on the accuracy of the self-force which can be computed. This may be mitigated somewhat by fitting for a large-m tail, but that fit itself requires more modes and is only ever approximate. Here, we propose a much better solution, using the higher-order terms in the singular field (those which have not been used in computing the effective source) to analytically deriving expressions for the tail. In many ways, this is analogous to the l-mode regularization scheme where there is a large-l tail and one can compute l-mode regularization parameters.

B. Derivation of m-mode Regularization Parameters

To derive analytic expressions for the large-m tail, we first note that an approximation to the singular field accurate to $\mathcal{O}(\epsilon^n)$ can be written in the form

$$\begin{aligned}\Phi^{\mathrm{S}}(x) &= \frac{1}{\rho_0^{2n+3}}\Big[\sum_{\substack{i=0\\ i\ \text{even}}}^{3(n+1)} A_{n,i}\sin^i(\Delta\phi/2) + \sum_{\substack{i=0\\ i\ \text{odd}}}^{3(n+1)} A_{n,i}\sin^{i-1}(\Delta\phi/2)\sin(\Delta\phi)\Big] + \mathcal{O}(\epsilon^{n+1}) \\ &= \frac{1}{\rho_0^{2n+3}}\Big[\sum_{\substack{i=0\\ i\ \text{even}}}^{3(n+1)} A_{n,i}\sin^i(\Delta\phi/2) + 2\sum_{\substack{i=0\\ i\ \text{odd}}}^{3(n+1)} A_{n,i}\sin^{i}(\Delta\phi/2)\cos(\Delta\phi/2)\Big] + \mathcal{O}(\epsilon^{n+1})\end{aligned} \tag{3.6}$$

where the coefficients $A_{n,i}$ are functions of position r_0 and θ_0, of the constants of motion E, L, C, and also of Δr and $\Delta\theta$. This form has the benefit of ensuring that the approximation is regular everywhere except on the world-line, while still being amenable to analytic integration in the ϕ direction. This makes it particularly appropriate for use in m-mode effective source calculations [64].

Using the leading orders (say, to $\mathcal{O}(\epsilon^p)$) in this expansion to compute an effective source, one is left with a singular field remainder which is finite, but of limited differentiability on the world-line. Since it is finite, we can safely set $\Delta r = \Delta\theta = 0$ in Eq. (3.6), leading to a singular field remainder which has the form

$$\Phi^{\mathrm{S}}(x) = \Big[2\Theta(\Delta\phi)-1\Big]\Big[\sum_{\substack{i=p+1\\ i\ \text{odd}}}^{n} B_{n,i}\sin^i(\Delta\phi/2) + 2\sum_{\substack{i=p+1\\ i\ \text{even}}}^{n} B_{n,i}\sin^i(\Delta\phi/2)\cos(\Delta\phi/2)\Big] + \mathcal{O}(\epsilon^{n+1}), \tag{3.7}$$

where $\Theta(\Delta\phi)$ is the Heaviside step function. Substituting this into Eq. (3.4) and noting that for even j

$$\int_{-\pi}^{\pi}\Big[2\Theta(\Delta\phi)-1\Big]\sin^j(\Delta\phi/2)\cos(\Delta\phi/2)e^{-im\phi}d\phi = \frac{2im}{j+1}\int_{-\pi}^{\pi}\Big[2\Theta(\Delta\phi)-1\Big]\sin^{j+1}(\Delta\phi/2)e^{-im\phi}d\phi \tag{3.8}$$

we are left with trivial integrals of the form

$$\int_{-\pi}^{\pi}\Big[2\Theta(\Delta\phi)-1\Big]\sin^{j+1}(\Delta\phi/2)e^{-im\phi}d\phi = \int_{-\pi}^{\pi}\Big[2\Theta(\Delta\phi)-1\Big]\sin^{j+1}(\Delta\phi/2)\cos(m\phi)d\phi. \tag{3.9}$$

As a result, we see that the real regularization parameters are given by the odd terms in the expansion of the singular field and the imaginary parameters are given by the even terms. Furthermore, we see that the falloff with m is always an even power of $1/m$ in the real part and an odd power of $1/m$ in the imaginary part.

While this analysis was done for the field, it should be noted that it equally well applies to the self-force. The only modification necessary is to compute the self-force from the singular field before setting $\Delta r = \Delta\theta = 0$; the remainder of the calculation proceeds in exactly the same way.

Finally, we note the m-mode regularization parameters derived in this way are dependent on the singular field being written in the form given in Eq. (3.6). Effective source calculation may use some other form for the approximation to the singular field (while still being accurate to the same order), in which case there is no guarantee that the regularization parameters given here are appropriate.

C. m-mode Regularization Parameters

Below, we give the results of applying this calculation to each of scalar, electromagnetic and gravitational cases in turn. In doing so, we omit the explicit dependence on m which in each case is

$$
\begin{aligned}
F^m_{a[2]} &= \frac{-4F_{a[2]}}{\pi(2m-1)(2m+1)}, \quad F^m_{a[4]} = \frac{24F_{a[4]}}{\pi(2m-3)(2m-1)(2m+1)(2m+3)},\\
F^m_{a[6]} &= \frac{-480F_{a[6]}}{\pi(2m-5)(2m-3)(2m-1)(2m+1)(2m+3)(2m+5)},\\
F^m_{a[8]} &= \frac{20160F_{a[8]}}{\pi(2m-7)(2m-5)(2m-3)(2m-1)(2m+1)(2m+3)(2m+5)(2m+7)}.
\end{aligned}
\tag{3.10}
$$

As the expressions for generic orbits of the Kerr spacetime are too large to be of use in printed form we give here only the representative expressions for two cases: the r component of the scalar self-force for a circular geodesic orbit and the quantity $H = \frac{1}{2}h_{ab}u^a u^b$ in the gravitational, circular orbits case. We direct the reader online [65] for more generic expressions in electronic form.

For circular orbits, the scalar self-force m-mode regularization parameters are:

$$
\begin{aligned}
F_{r[2]} = {}& \frac{M}{24r_0^4[aM + r_0\sqrt{Mr_0}][2a\sqrt{Mr_0} + r_0(r_0-3M)]^{3/2}[a^2 + r_0(r_0-2M)]^{3/2}}\Big[24a^7M^2\\
&- 24a^6M\sqrt{Mr_0}(M-2r_0) - 4a^5Mr_0(23M^2 + Mr_0 - 6r_0^2) + 2a^4Mr_0\sqrt{Mr_0}(45M^2 - 112Mr_0 + 31r_0^2)\\
&+ 2a^3Mr_0^2(45M^3 + 45M^2r_0 - 73Mr_0^2 + 19r_0^3) - 3a^2r_0^2\sqrt{Mr_0}(29M^4 - 88M^3r_0 + 38M^2r_0^2 - 4Mr_0^3 + r_0^4)\\
&- 6aMr_0^4(29M^3 - 43M^2r_0 + 21Mr_0^2 - 3r_0^3) - 3r_0^5\sqrt{Mr_0}(29M^3 - 25M^2r_0 + 3Mr_0^2 + r_0^3)\Big],
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
F_{r[4]} = {}& \frac{M^2}{1440r_0^9[aM + r_0\sqrt{Mr_0}][2a\sqrt{Mr_0} + r_0(r_0-3M)]^{7/2}[a^2 + r_0(r_0-2M)]^{3/2}}\Big[-23040a^{14}M^2\sqrt{Mr_0}\\
&+ 11520a^{13}M^2r_0(M-8r_0) + 384a^{12}Mr_0\sqrt{Mr_0}(461M^2 - 81Mr_0 - 360r_0^2) - 192a^{11}Mr_0^2(307M^3\\
&- 3780M^2r_0 + 1233Mr_0^2 + 480r_0^3) - 64a^{10}r_0^2\sqrt{Mr_0}(8549M^4 - 3593M^3r_0 - 16212M^2r_0^2\\
&+ 6336Mr_0^3 + 360r_0^4) + 32a^9Mr_0^3(2835M^4 - 69401M^3r_0 + 46565M^2r_0^2 + 13779Mr_0^3 - 8748r_0^4)\\
&+ 192a^8r_0^3\sqrt{Mr_0}(4470M^5 - 3621M^4r_0 - 15645M^3r_0^2 + 12662M^2r_0^3 - 1529Mr_0^4 - 342r_0^5)\\
&+ 16a^7r_0^4(-1479M^6 + 210966M^5r_0 - 224760M^4r_0^2 - 49213M^3r_0^3 + 93619M^2r_0^4 - 19953Mr_0^5 + 180r_0^6)\\
&- 16a^6r_0^4\sqrt{Mr_0}(43101M^6 - 61443M^5r_0 - 271980M^4r_0^2 + 343776M^3r_0^3 - 100489M^2r_0^4 - 7763Mr_0^5\\
&+ 4158r_0^6) + 12a^5r_0^5(-2367M^7 - 221220M^6r_0 + 337457M^5r_0^2 + 71894M^4r_0^3 - 262111M^3r_0^4 + 111498M^2r_0^5\\
&- 14459Mr_0^6 + 588r_0^7) + 3a^4r_0^5\sqrt{Mr_0}(76125M^7 - 176307M^6r_0 - 1157559M^5r_0^2 + 1949709M^4r_0^3\\
&- 855873M^3r_0^4 - 26505M^2r_0^5 + 76235Mr_0^6 - 8065r_0^7) + 12a^3r_0^7(76125M^7 - 152637M^6r_0 - 93174M^5r_0^2\\
&+ 281414M^4r_0^3 - 166063M^3r_0^4 + 36555M^2r_0^5 - 3480Mr_0^6 + 460r_0^7) + 18a^2r_0^8\sqrt{Mr_0}(76125M^6\\
&- 145182M^5r_0 + 70771M^4r_0^2 + 16696M^3r_0^3 - 19905M^2r_0^4 + 3190Mr_0^5 + 225r_0^6) + 36ar_0^{10}(25375M^6\\
&- 47369M^5r_0 + 31856M^4r_0^2 - 8692M^3r_0^3 + 705M^2r_0^4 - 75Mr_0^5 + 40r_0^6)\\
&+ 9r_0^{11}\sqrt{Mr_0}(25375M^5 - 47015M^4r_0 + 29014M^3r_0^2 - 4814M^2r_0^3 - 1365Mr_0^4 + 405r_0^5)\Big].
\end{aligned}
\tag{3.12}
$$

and the gravitational, m-mode parameters for H are

$$
\begin{aligned}
H_{[2]} = {}& \sqrt{\frac{M\left(a^2 + r_0(r_0-2M)\right)}{\left(2a\sqrt{M} + \sqrt{r_0}(r_0-3M)\right)}} \times\\
&\frac{44a^4M + 88a^3\sqrt{Mr_0^3} - 3a^2r_0(M-r_0)(29M+15r_0) + 6a\sqrt{M}r_0^{5/2}(14r_0 - 29M) - 87Mr_0^4 + 45r_0^5}{12r_0^{15/4}\left(a + \sqrt{\frac{r_0^3}{M}}\right)\left(a^2 + r_0(r_0-2M)\right)}.
\end{aligned}
\tag{3.13}
$$

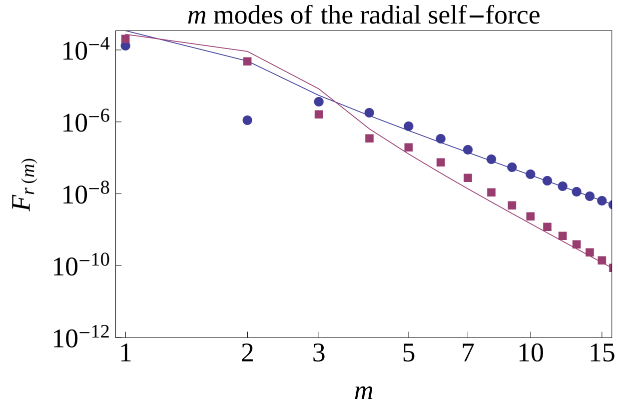

FIG. 2. Regularization of the radial component of the self-force for the case of a scalar particle on a circular geodesic of radius $r_0 = 10M$ in Kerr spacetime with $a = 0.6M$. The numerical self-force modes (blue dots) asymptotically match the $F^m_{r[4]}$ regularization parameter (blue line) for large m. After regularization, the remainder (purple squares) fall off as m^{-6}.

D. Example

As an example application of these m-mode regularization parameters, we consider the case of a scalar charge on a circular geodesic orbit of radius $10M$ in the Kerr spacetime with $a = 0.6M$. The self-force in this case was computed in Ref. [64] using the m-mode effective source approach with an effective source derived from an approximation to the singular field of the form (3.6) accurate to $\mathcal{O}(\epsilon^2)$. As expected, this gave numerical results for the m-modes of the self-force which asymptotically fall off as m^{-4}. In this case, the $F^m_{r[2]}$ parameter is not needed as it has already been subtracted through the effective source calculation. However, the $F^m_{r[4]}$ parameter has not been subtracted and asymptotically gives the leading order behaviour (in $1/m$) of the modes. Subtracting this from the numerical results therefore leave a remainder which falls off as m^{-6}. Furthermore, a numerical fit of this remainder can be done to numerically determine the next two parameters, in this case giving $F_{r[6]} = 0.108797$ and $F_{r[8]} = 11.3398$.

In Fig. 2, we plot the results of subtracting the $F^m_{r[4]}$ and $F^m_{r[6]}$ regularization parameters in turn from the raw numerical data. For large m, the numerical data (blue dots) falls of as m^{-4}, with the coefficient matching our analytic prediction given by $F^m_{r[4]}$ (blue line). Subtracting this leading order behaviour, we find that the remainder falls off as m^{-6} (purple squares and line) and then m^{-8} (brown diamonds and line), as expected.

IV. DISCUSSION

This paper extends the work of Paper I to the case of equatorial geodesic orbits in the Kerr spacetime. However, this only reflects a subset of the possible geodesic orbits in that case. In general, geodesics of the Kerr spacetime do not lie in the equatorial plane. While our calculation could be extended to cover the case of these more generic geodesics, we have chosen here to restrict ourselves to the case of equatorial motion and work with the significantly simpler expressions that ensue, leaving the more general case for future work.

In our analysis, we have made use of scalar spherical harmonics which are not particularly well suited to the Kerr spacetime and gravitational perturbations. A more appropriate choice of basis functions may be the spheroidal harmonics; it may be more sensible to compute regularization parameters for a spheroidal harmonic basis. However, from a practical perspective, most existing numerical self-force calculations already make use of lower order versions of the expressions given here. The example clearly shows that these existing calculations gain significant improvements in accuracy from the use of spherical harmonic expansions. In this way, the end justifies the means: despite not being a natural choice, the use of spherical harmonics is a good practical choice. Nevertheless, an adaptation of our calculation to the spheroidal harmonic basis and to other gauges would make the results applicable in a much wider range of contexts.

The Lorenz gauge metric perturbation equation on Kerr spacetime has yet to be been shown to be fully separable. It is likely that this would require the development of tensor spheroidal harmonics, which are as-yet unknown. In the absence of these, it has not been possible to test the validity of our electromagnetic and gravitational l-mode regularization parameters. However, as in Paper I, deriving the expressions by independent methods gives us strong confidence in our results. Another check is to set $a = 0$, in which case the results agree with those of Paper I, which

we know to be correct.

Note that the m-mode scheme is not affected by this issue as it is equally applicable to both Schwarzschild and Kerr spacetimes and is likewise equally as valid in the gravitational, electromagnetic and scalar cases. The only caveat is that the m-mode regularization parameters are only guaranteed to be correct for an effective source derived from a compatible approximation to the singular field. Since there is a large amount of flexibility in the effective source approach, if one chooses an incompatible singular field approximation, the regularization parameters here must be modified appropriately.

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